Math 137: Algebraic Geometry

Spring 2022

Problem set #7

due Wednesday, March 30 at noon

Throughout, K is assumed to be an algebraically closed field.

Problem 1. Which of the following morphisms are finite? (Say for $K = \mathbb{C}$.)

a) The morphism $K^2 \to K$ sending (x, y) to $x^3y + xy^3 + 3x + 1$.

b) The morphism $K \to K^2$ sending x to (x^2, x^3).

Problem 2.

a) Let $\varphi : V \to W$ be a morphism. Show that if V is the union of algebraic subsets $V_1, \ldots, V_n$ and each restriction $\varphi : V_i \to W$ is a finite morphism, then φ is a finite morphism.

b) Let $V \subseteq K^n$ be a finite set and let W be any algebraic set. Show that every map $\varphi : V \to W$ is a finite morphism.

Problem 3. Let $\varphi : V \to W$ be a dominant morphism between irreducible algebraic sets. Assume $\Gamma(V)$ is generated by n elements as a $\varphi^*(\Gamma(W))$-module. Show that the preimage of any point $Q \in W$ has size at most n.

Problem 4.

a) Let L be a finitely generated field extension of K with $n = \operatorname{trdeg}(L|K)$ and let $R \subseteq L$ be a finitely generated ring extension of K whose field of fractions is L. Show that there are elements $a_1, \ldots, a_n$ of R such that R is an integral ring extension of $K[a_1, \ldots, a_n]$.
Hint: Translate this into a geometric statement.

b) (bonus) Show the statement in a) without assuming that K is algebraically closed.

Problem 5. Say $K = \mathbb{C}$. Construct a surjective but nonfinite morphism $\varphi : V \to W$ between irreducible algebraic sets such that every $P \in W$ has only finitely many preimages. (You get half the points if V is reducible.)

Reminder: You can still submit problems 6 and 7 from problem set 6.