

Math 137: Algebraic Geometry

Spring 2022

Problem set #3

due Wednesday, February 16 at noon

Throughout, K is assumed to be an algebraically closed field.

Problem 1 (Déjà vu). Let A be an irreducible algebraic subset of K^n and let B be an irreducible algebraic subset of K^m . Show that the cartesian product $A \times B$ is an irreducible algebraic subset of $K^n \times K^m = K^{n+m}$.

Problem 2. Show Corollary 4.5.6: If S is an integral ring extension of R and T is an integral ring extension of S , then T is an integral ring extension of R .

Problem 3. Show that any integral ring-finite ring extension is module-finite.

Problem 4. Find a ring-finite ring extension S of a ring R for which S is a field, but which is not module-finite.

Problem 5. a) What is the integral closure of $K[X^2, X^3] \subseteq K[X]$ in $K(X)$?

b) Let $R = \mathbb{Z}/4\mathbb{Z}$. What is the integral closure of R in $R[X]$?

Problem 6. Let X be a topological space. Let $A \neq \emptyset$ be a closed subset and equip it with the subspace topology. Show that the following are equivalent:

- a) We cannot write $A = A_1 \cup A_2$ with closed subsets $A_1, A_2 \subsetneq A$.
- b) Any two open subsets $\emptyset \neq U_1, U_2 \subseteq A$ (for the subspace topology) intersect.
- c) Every open subset $\emptyset \neq U \subseteq A$ (for the subspace topology) is dense in A .

Such a set A is called *irreducible*. (Note that the irreducible algebraic subsets of K^n are exactly the irreducible closed subsets for the Zariski topology.)

Problem 7. Let $f : X \rightarrow Y$ be a continuous map between topological spaces. Let $A \subseteq X$ be a closed irreducible subset. Show that $\overline{f(A)} \subseteq Y$ is irreducible.