4. Nullstellensätze

German for: root theorems

4.1. Nichtnullstellensatz

German for: non-root theorem

Thm 4.1.1

Assume that K is infinite.

Then, $I(K^n) = 0$.

(In other words, there is no pol. $0 \neq f \in K[x_1, \dots, x_n]$ that vanishes everywhere.)

Rmk This is wrong for finite fields K:

$f(x_1, \dots, x_n) := \prod_{a \in K} (x_1 - a)$ vanishes everywhere.

Pf of Thm Use induction over n.

($n=0$: silly)

$n=1$: nonzero pol. have only finitely many roots

$n-1 \Rightarrow n$: Let $0 \neq f \in K[x_1, \dots, x_n]$.

Write $f(x_1, \dots, x_n) = \sum_{i=0}^{d} g_i(x_1, \dots, x_{n-1}) x_n^i$

with $g_i \in K[x_1, \dots, x_{n-1}]$, $g_d \neq 0$.

By induction, we have

$g_d(a_1, \dots, a_{n-1}) \neq 0$ for some $(a_1, \dots, a_{n-1}) \in K^{n-1}$.

$\Rightarrow 0 \neq f(a_1, \dots, a_{n-1}, x_n) \in K[x_n]$ (pol. of degree d)

By the $n=1$ case, we then have

$f(a_1, \dots, a_{n-1}, a_n) \neq 0$ for some $a_n \in K$. $\square$

4.2. Weak Nullstellensatz

Thm 4.2.1 (Weak Nullstellensatz)

German for: root theorem

Assume that K is algebraically closed. For any ideal $I \subsetneq K[x_1,\dots,x_n]$, we have $V(I) \neq \emptyset$.

Pf soon...

Rmk This is false if K is not algebraically closed.

Pf If K isn't alg. cl., there is a nonconstant pol. $f \in K[x]$ without roots.

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$(f) \subsetneq K[x]$ $\qquad$ $V((f)) = \emptyset$. $\square$

4.3 Hilbert's Nullstellensatz

Given an ideal $I \subseteq k[x_1, \dots, x_n]$, what is $\mathcal{I}(\mathcal{V}(I))$?
When is $\mathcal{I}(\mathcal{V}(I)) = I$?

Ex $I = (x^2(x-1)(x-2)^3) \subseteq \mathbb{R}[x]$
$\Rightarrow \mathcal{V}(I) = \{0, 1, 2\}$
$\Rightarrow \mathcal{I}(\mathcal{V}(I)) = (x(x-1)(x-2)) \subseteq \mathbb{R}[x]$ is larger than I.

Note If $f^n \in I$ for some $n \geq 1$, then $f \in \mathcal{I}(\mathcal{V}(I))$.

Pf If $P \in \mathcal{V}(I)$, then $f(P)^n = 0$.
$\Rightarrow f(P) = 0 \quad \| \Rightarrow f \in \mathcal{I}(\mathcal{V}(I))$. $\square$

Def The radical of an ideal I of any ring R is the set
$\mathrm{Rad}(I) := \sqrt{I} := \{ f \in R \mid f^n \in I \text{ for some } n \geq 1\}$.

Lemma 4.1 $\sqrt{I}$ is an ideal.

Pf • Let $f, g \in \sqrt{I}$.
$\Rightarrow f^n \in I,\ g^m \in I$ for some $n, m \geq 1$.

$$\Rightarrow (f+g)^{n+m} = \sum_{\substack{i,j \geq 0:\\ i+j=n+m}} \binom{n+m}{i} \cdot \underbrace{\underbrace{f^i}_{\substack{\in I\\ \text{for } i \geq n}} \cdot \underbrace{g^j}_{\substack{\in I\\ \text{for } j \geq m}}}_{\substack{\in I\\ \text{always}}} \in I$$

$\Rightarrow f+g \in \sqrt{I}$

- Let $f \in \sqrt{I}$, $g \in R$.

  $\Rightarrow f^n \in I$ for some $n \geq 1$

  $\Rightarrow (af)^n = a^n f^n \in I$

  $\Rightarrow af \in \sqrt{I}$

- Clearly, $0 \in \sqrt{I}$. $\square$

<u>Def</u> An ideal I is a <u>radical ideal</u> if $\sqrt{I} = I$.

<u>Rmk</u> $\sqrt{I}$ is a radical ideal: $\sqrt{\sqrt{I}} = \sqrt{I}$.

<u>Rmk</u> If R is a unique factorization domain and we have a factorization $f = u \cdot g_1^{e_1} \cdots g_r^{e_r}$, then $\sqrt{(f)} = (g_1 \cdots g_r)$.

Thm 4.3.2 (Hilbert's Nullstellensatz)

Assume that K is algebraically closed.
Then, $\mathcal{J}(\mathcal{V}(I)) = \sqrt{I}$ for any ideal I of $K[X_1, \dots, X_n]$.

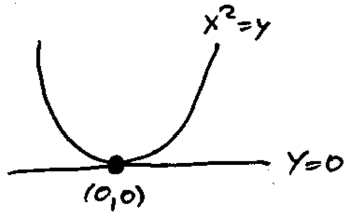
Ex. If $n=1$, $I=(f)$ with
$f = c(X-a_1)^{e_1} \cdots (X-a_r)^{e_r}$, then

$\mathcal{V}(I) = \{a_1, \dots, a_r\}$,
$\mathcal{J}(\mathcal{V}(I)) = ((X-a_1)\cdots(X-a_r)) = \sqrt{(f)}$.

~~Rmk The Thm is wrong if K is not alg. closed.~~

~~Let f ∈ K[X]~~

Ex. $n=2$, $I = (X^2-Y, Y) = (X^2, Y)$



$\Rightarrow \mathcal{V}(I) = \{(0,0)\}$

$\Rightarrow \mathcal{J}(\mathcal{V}(I)) = (X, Y) = \sqrt{I}$

Rmk Hilbert's Nsts (for k) $\Rightarrow$ Weak Nsts (for k)

(In part., Hilbert's Nsts fails for fields that aren't alg. closed)

Pf If $\mathcal{V}(I)=\emptyset$, then $\underbrace{\mathcal{J}(\mathcal{V}(I))}_{=\sqrt{I}} = K[x_1,\dots,x_n] \ni 1$.

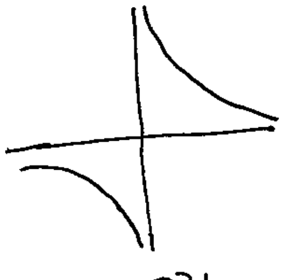
$\Rightarrow \underset{1=}{1^n} \in I$ for some $n \geq 1$. $\square$

Rmk Hilbert's Nsts $\Rightarrow$ Nichtnullstellensatz

Pf $\mathcal{J}(K^n) = \mathcal{J}(\mathcal{V}(0)) = \sqrt{0} = 0$. $\square$

Preparation

$\mathbb{R}\setminus\{0\}$ isn't an alg. subset of $\mathbb{R}$



$\{(x,y)\in\mathbb{R}^2 \mid xy=1\}$ is an alg. subset of $\mathbb{R}^2$ and its projection onto the x-axis is $\mathbb{R}\setminus\{0\}$.

Pf of Hilbert's Nsts (assuming Weak Nsts)

"$\mathcal{I}(\mathcal{V}(I)) \supseteq \sqrt{I}$": done earlier

"$\mathcal{I}(\mathcal{V}(I)) \subseteq \sqrt{I}$":

Let $f \in \mathcal{I}(\mathcal{V}(I))$.

$\Rightarrow \forall P \in \mathcal{V}(I): \ f(P) = 0$

$\Rightarrow \{P \in \mathcal{V}(I) \mid f(P) \neq 0\} \overset{\subseteq K^n}{=} \emptyset$

We have a bijection

$$\{P \in \mathcal{V}(I) \mid f(P) \neq 0\} \longleftrightarrow \{(P,t) \in \underbrace{\mathcal{V}(I) \times K}_{\subseteq K^{n+1}} \mid f(P) \cdot t = 1\} = \underset{\substack{\cap \\ K^{n+1}}}{\mathcal{V}(I')},$$

where $I' \subseteq K[X_1, \dots, X_n, T]$ is the ideal generated by the elements of I and by the polynomial $f(x_1, \dots, x_n) \cdot T - 1$.

$LHS = \emptyset \Rightarrow \underset{\substack{\shortparallel \\ \mathcal{V}(I')}}{RHS} = \emptyset$

$\underset{\substack{\uparrow \\ \text{Weak Nsts}}}{\Longrightarrow} \quad I' = K[X_1, \dots, X_n, T] \ni 1$

$\Rightarrow$ We can write 1 as a lin. comb. of el. of I and $f(x_1,\dots,x_n)\cdot T - 1$, with coefficients in $K[x_1,\dots,x_n,T]$.

$\Rightarrow$ We can write

$$1 = \sum_{i=0}^{d} p_i(x_1,\dots,x_n)\cdot T^i + \big(f(x_1,\dots,x_n)\cdot T - 1\big)\cdot q(x_1,\dots,x_n,T)$$

with $p_i \in I$, $q \in K[x_1,\dots,x_n,T]$.

That's an eq. in $K[x_1,\dots,x_n,T] \subseteq K(x_1,\dots,x_n)[T]$.

Plug in $T = \frac{1}{f(x_1,\dots,x_n)}$:

$$1 = \sum_{i=0}^{d} p_i(\dots)\cdot\frac{1}{f(\dots)^i} \quad \text{in } K(x_1,\dots,x_n)$$

$$\Rightarrow f^d = \sum_{i=0}^{d} \underbrace{p_i}_{\in I}\, \underbrace{f^{d-i}}_{\in K[x_1,\dots,x_n]} \in I$$

$\Rightarrow f \in \sqrt{I}$. $\square$