Math 223b: Algebraic Number Theory

Spring 2021

Problem set #9

due Friday, April 16 at noon

We assume that K is a field of characteristic zero throughout this problem set.

Problem 1. Let E be an elliptic curve of rank r over $\mathbb{Q}$. Let $\varphi : E \to \mathbb{P}^n_{\mathbb{Q}}$ be any nonconstant morphism. Show that the limit

$$\lim_{T\to\infty} \frac{\#\{P \in E(\mathbb{Q}) \mid h(\varphi(P)) \leqslant T\}}{T^{r/2}}$$

exists and is positive.

Problem 2. Show that $\mathbb{P}^n_K$ cannot be given the structure of a group variety (for $n \geqslant 1$).

Problem 3. a) Let $V \subseteq \mathbb{A}^n_K$ and $W \subseteq \mathbb{A}^m_K$ be geometrically irreducible. Show that $V \times W \subseteq \mathbb{A}^{n+m}_K$ is geometrically irreducible.

b) Construct affine varieties $V \subseteq \mathbb{A}^n_{\mathbb{Q}}$ and $W \subseteq \mathbb{A}^m_{\mathbb{Q}}$ which are irreducible over $\mathbb{Q}$, but such that $V \times W$ is not irreducible over $\mathbb{Q}$.

Problem 4. Let G be a group variety and let $A \subseteq G$ be a subgroup which is an abelian variety. Show that A is contained in the center of G.

Problem 5. Consider any continuous action of $\mathrm{Gal}(\overline{K}|K)$ on a finite group G. Show that there is a group variety G' over K such that the elements of $G'(\overline{K})$ are in bijection with the elements of G and the action of $\mathrm{Gal}(\overline{K}|K)$ on G corresponds under this bijection to the natural action of $\mathrm{Gal}(\overline{K}|K)$ on $G'(\overline{K})$. (Hint: Consider the subring of $\prod_{g\in G} \overline{K}$ fixed by a suitable action of $\mathrm{Gal}(\overline{K}|K)$.)