# Math 223b: Algebraic Number Theory

## Spring 2021

### Problem set #10

due Wednesday, April 28 at noon

We assume that $K$ is a field of characteristic zero throughout this problem set.

**Problem 1.** Show that the $r$-th symmetric power of $\mathbb{A}^n$ is singular for all $r, n \geqslant 2$. (Hint: You don't need to completely determine the quotient variety for this problem!)

**Problem 2.** Let $G$ be a finite group variety defined over $K$ acting on an affine variety $V$ defined over $K$ such that the map $G \times V \to V$ sending $(g, x)$ to $gx$ is a morphism defined over $K$. Show that the quotient variety $V^G$ and the quotient map $\pi : V \to V^G$ are also defined over $K$. (Hint: Consider the action of $\mathrm{Gal}(\overline{K}|K)$ on the polynomials defining $V^G$ and $\pi$.)

**Problem 3.** Prove Lemma 3.3.11: Assume $K$ is algebraically closed, $V \subseteq \mathbb{A}^n_K$ is irreducible and $v_0 \in V$ a nonsingular point. Let $S \subseteq V$ be the union of the irreducible subvarieties $W \subsetneq V$ containing $v_0$ which are nonsingular at $v_0$. Then, $\overline{S} = V$. (Hints: Only consider intersections of $V$ with hyperplanes. Use the fact that the tangent cone $T$ of a variety $A$ at any point $a \in A$ satisfies $\dim(T) = \dim(A)$.)

**Problem 4.** Let $\varphi : V \to W$ be a morphism between irreducible affine varieties $V, W$ over $K$. Assume that for every field extension $L$ of $K$, there is a dense open subset $U \subseteq W_L$ such that each $Q \in U(L)$ has exactly one preimage $P \in V_L(L)$. Show that there is a rational map $\tau : W \dashrightarrow V$ such that $\varphi \circ \tau = \mathrm{id}$ on a dense open subset $U' \subseteq W$.

**Problem 5** (bonus; if you know representation theory of finite groups)**.** Let $G$ be a finite group acting on $\mathbb{A}^n_K = K^n$ via the representation $\rho : G \to \mathrm{GL}_n(K)$. For $d \geqslant 0$, let $a_d$ be the dimension of the space of homogeneous polynomials of degree $d$ in the ring of invariants $\Gamma(\mathbb{A}^n_K)^G \subseteq K[X_1, \ldots, X_n]$. Show the following identity of formal power series in $Z$:

$$\sum_{d \geqslant 1} a_d Z^d = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - Z\rho_g)}.$$