Rmk In the def. of complete varieties V, we could allow arbitrary (not just affine) varieties W.

Pf Let $W_1, \dots, W_n$ be the affine patches of W.

$$W = \bigcup_i W_i \quad , \quad W_i \subseteq W \text{ open.}$$

Consider a closed subset A of $V \times W$ and the proj. $\pi : V \times W \to W$.

$\pi(A \cap (V \times W_i))$ is closed in W_i.

$$\Rightarrow \pi(A) = \bigcap_i \underbrace{\left(\underbrace{\pi(A \cap (V \times W_i))}_{cl. \subseteq W_i} \cup \underbrace{(W \setminus W_i)}_{cl. \subseteq W} \right)}_{cl. \subseteq W}$$

is closed in W. $\square$

Lemma 3.1.3 If $\varphi : V \to W$ is a morphism and V is complete, then $\varphi(V)$ is closed in W and complete.

Cor 3.1.4 φ is closed

Pf If $A \subseteq V$ is closed, A is also complete. Apply the lemma to the restriction $\varphi : A \to W$. $\square$

Pf of Lemma

$\varphi(V)$ closed: Consider the graph of φ:

$$\{(v,w) \in V \times W \mid w = \varphi(v)\}$$

It's a closed subset of $V \times W$.

Its image under the proj. $V \times W \to W$ is $\varphi(V)$.

$\varphi(V)$ complete

W.l.o.g. φ is dominant, and hence surjective (over $\overline{K}$). $W = \varphi(V)$

Let Z be an affine var. and $A \subseteq W \times Z$ closed

$$V \times Z \xrightarrow{\psi} W \times Z \xrightarrow{\pi} Z$$
$$(v,z) \mapsto (\varphi(v), z)$$

$$\pi(A) = \pi \circ \psi(\underbrace{\psi^{-1}(A)}_{\text{closed}})$$ is closed in Z because

V is complete. $\square$

Lemma 3.1.5 An affine variety $V \subseteq \mathbb{A}^n_K$ is complete if and only if $\# V(\overline{K}) < \infty$.

Pf "$\Leftarrow$" clear

"$\Rightarrow$" Consider the projection $\varphi_i : V \to \mathbb{A}^1_K$.
$(x_1, \dots, x_n) \mapsto x_i$

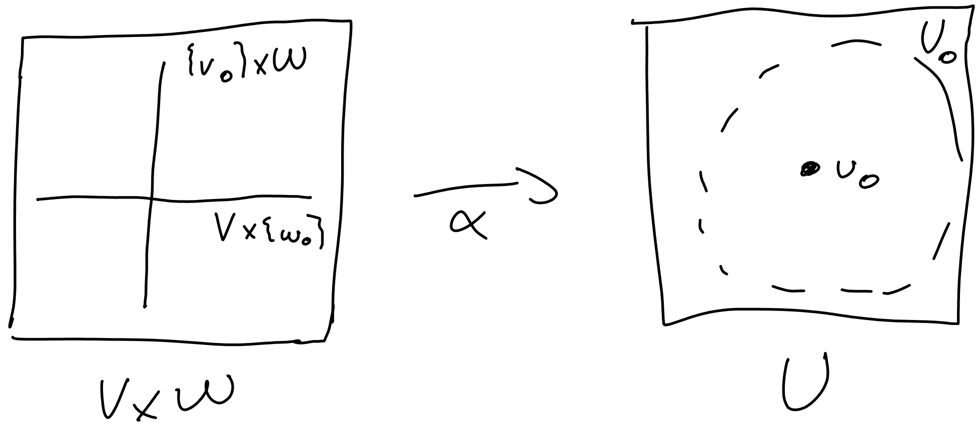
By the lemma, the image $\varphi_i(V)$ is closed, complete in $\mathbb{A}^1_K$.
$\Rightarrow \# \varphi_i(V) < \infty \ \forall i$ $\square$

Thm 3.1.6 (Rigidity theorem)

Let V be complete and W, U be arbitrary var.

Assume that $V \times W$ is irred.

Let $\alpha : V \times W \to U$ be a morphism such that $\alpha(V \times \{w_0\}) = \alpha(\{v_0\} \times W) = \{u_0\}$ for some $v_0 \in V(k)$, $w_0 \in W(k)$, $u_0 \in U(k)$.



Then, $\alpha(V \times W) = \{u_0\}$.

Pf Let $u_0 \in U_0 \subseteq U$ be an affine patch.

Then, $Z := \{w \in W \mid \exists v \in V : \alpha(v,w) \notin U_0\}$

is the image of the closed set

$\alpha^{-1}(U \setminus U_0) \subseteq V \times W$ under the proj.

$V \times W \to W$.

$\Rightarrow Z \subseteq W$ is closed

V complete

$\Rightarrow w_0 \in W \setminus Z \subseteq W$ is open

For any $w \in W \setminus Z$, consider the morphism

$$\begin{aligned} V &\longrightarrow U_0 . \\ v &\longmapsto \alpha(v,w) \end{aligned}$$

Its image is complete and affine, hence finite. The image contains $\alpha(v_0, w) = v_0$. Since V is irred. (because $V \times W$ is), the image is also irred. Hence, the image is $\{v_0\}$.

$\Longrightarrow \alpha(v,w) = v_0 \quad \forall v \in V, w \in W \setminus Z$

But $V \times (W \setminus Z)$ is a nonempty open subset of the irred. var $V \times W$, hence dense.

$\Longrightarrow \alpha(v,w) = v_0 \quad \forall v \in V, w \in W.$ $\square$

3.2. Basic properties of algebraic groups

Lemma 3.2.1 Let G be an alg.-group.

The connected component G_0 of G containing the identity $e \in G$ is a normal subgroup of G. The quotient G/G_0 is the (finite) group of connected components of G.

Pf Let A, B be conn. comp. of G.

$\Rightarrow$ $A \times B \subseteq G \times G$ is connected

$\Rightarrow$ Its image $A \cdot B$ under the morphism

$$G \times G \longrightarrow G$$
$$(g, h) \longmapsto gh$$

is connected.

$\Rightarrow$ $A \cdot B$ is contained in some conn. comp.

Similarly, A^{-1} is contained in some conn. comp.

$\Rightarrow$ We obtain a (well-def!) group law on $S := \{\text{conn. comp.}\}$ with a group hom. $f : G \twoheadrightarrow S$

$$g \longmapsto \text{conn. comp. containing } g$$
$$e \longmapsto G_0$$

Then, $G_0 = \ker(f)$. $\square$

<u>Lemma 3.2.2</u> Any connected alg. group G is irreducible and in fact smooth.

<u>Pf</u> Say $G = V_1 \cup \ldots \cup V_r$ is the decomposition into irred. comp.

<u>Rmk</u> Any conn. (over K) alg. group G is geometrically (over $\overline{K}$) connected.

<u>Pf</u> $Gal(\overline{K}|K)$ acts transitively on the conn. comp. of $G(\overline{K})$ over $\overline{K}$. But $e \in G(K)$ is fixed by every el. of $Gal(\overline{K}|K)$ and lies in just one conn. comp. $\square$

$\rightsquigarrow$ w.l.o.g. $K = \overline{K}$.

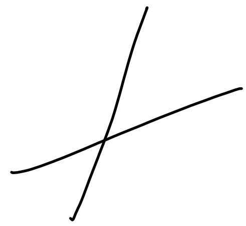
For any $g \in G(\overline{K})$, the translation morphism

$$\tau_g : G \to G, \quad h \mapsto gh$$

is an isomorphism of varieties and hence permutes the irred. comp.

G irred: Consider the alg. set

$$S = \bigcup_{i \neq j} (V_i \cap V_j) \subseteq G(\overline{k}).$$



$$\Rightarrow \underbrace{\tau_g(S)}_{= g \cdot S} = S \quad \forall g \in G(\overline{k})$$

$$\Rightarrow S = \emptyset \quad \text{or} \quad S = G(\overline{k})$$

$S = \emptyset \Rightarrow V_i \cap V_j = \emptyset \ \forall i,j \xRightarrow{G \text{ conn}}$ just one irred. comp.

$S = G(\overline{k})$: impossible because

$$\dim(S) \leq \max_{i \neq j} \dim(V_i \cap V_j) < \max_i \dim(V_i) = \dim(G)$$

G smooth: Let $S' \subseteq G(\overline{k})$ be the set of singular points. $\Rightarrow gS' = \tau_g(S') = S'$

$$\Rightarrow S' = \emptyset \quad \text{or} \quad S' = G(\overline{k})$$

$S' = G(\overline{k})$: impossible by problem 2b from problem set 2.

$\square$

Def A homomorphism of alg. groups is a group hom. which is also a morphism.

Rmk The kernel of a hom. of alg. groups is an alg. group.

Lemma 3.2.3 The image of a hom. $\varphi: G \to H$ of alg. groups is a closed subgroup of H.

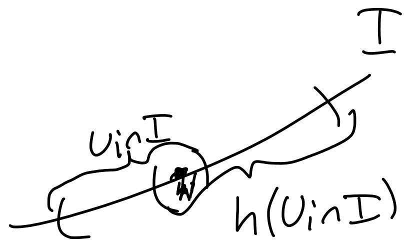
Pf Let $I = \overline{\varphi(G)}$.

W.l.o.g. G is connected (...), so G and I are irred. Of course $\varphi(G)$ is a subgr. of H. $\Rightarrow$ By continuity of mult., inverse, the closure I is also a subgroup of H.

By Chevalley's theorem, $\varphi(G) \subseteq H$ is locally closed, so $\varphi(G) = \bigcup_{i=1}^{n} (U_i \cap T_i)$ for open $U_i \subseteq G$ and closed $T_i \subseteq I$ with $U_i \cap T_i \neq \emptyset$. Since $I = \overline{\varphi(G)} \subseteq \bigcup_i T_i$ is irred., we have $T_i = I$ for some i.

$\Rightarrow \varphi(G) \supseteq U_i \cap I \neq \emptyset$

Take any $h \in I$. Then, both



$$U_i \cap I$$

and

$$h \cdot (U_i \cap I)$$

are nonempty open subsets of I.

Since I is irred., they intersect.

Take $h' \in (U_i \cap I) \cap (h \cdot (U_i \cap I))$.

$$\subseteq \varphi(G) \cap (h \cdot \varphi(G)).$$

$$\Rightarrow \exists\, g, g' \in G : \varphi(g) = h' = h \cdot \varphi(g')$$

$$\Rightarrow h = \varphi(g)\varphi(g')^{-1} = \varphi(g g'^{-1}) \in \varphi(G).$$

$$\Rightarrow \varphi(G) = I = \overline{\varphi(G)}. \qquad \square$$