Rmk $L(D + \mathrm{div}(g)) \cong L(D)$ for any $g \in K(C)^\times$,

$$f \mapsto fg$$

so $\ell(D)$ only depends on the divisor class of D in $\mathrm{Cl}(C)$.

Ex Let $C = \mathbb{P}^1_K$ and let $D \in \mathrm{Div}(C)$ of degree d.
Then, $\ell(D) = \begin{cases} 0, & d < 0, \\ d+1, & d \geq 0. \end{cases}$

Pf Let $d \geq 0$. w.l.o.g. $D = d \cdot \underbrace{[1:0]}_{\infty}$.

$\Rightarrow L(D) = \{ f \in K(C) \mid f$ has at most a pole of order d at $[1:0]$ and no other poles$\}$

$$= \left\{ \frac{f(X,Y)}{Y^d} \;\middle|\; f(X,Y) \text{ homogeneous of degree } d \right\}.$$

$\square$

Rmk $L(0) = \mathcal{O}_C(C) = K$

Rmk $L(D) = 0$ if $\deg(D) < 0$

Pf $\deg(\mathrm{div}(f) + D) = \deg(D) < 0$.

$\Rightarrow \mathrm{div}(f) + D \not\geq 0$. $\square$

Rmk $L(D) = 0$ if $\deg(D) = 0$ but $[D] \neq 0$ in $\mathcal{Cl}(C)$.

Pf let $f \neq 0$. $\deg(\mathrm{div}(f) + D) = \deg(D) = 0$

$\Rightarrow$ If $\mathrm{div}(f) + D \geq 0$, then $\mathrm{div}(f) + D = 0$.

$\square$

Rmk $L(D) \subseteq L(D')$ if $D \leq D'$.

(Important) Rmk $\ell(D)$ doesn't depend on the base field K: If we denote the corresponding $\bar{K}$-vector space of rational functions defined over $\bar{K}$ by $\bar{L}(D) \subseteq K(C) \otimes_K \bar{K}$, then $\bar{L}(D) = L(D) \otimes_K \bar{K}$.

Lemma $\ell(D) - 1 \leq \ell(D-P) \leq \ell(D)$ $\forall D \in \mathrm{Div}(C), P \in C(\bar{K})$

Cor $\ell(D) < \infty$.

Pf of Lemma Let $D = \sum_Q n_Q Q$. The the linear map $L(D) \longrightarrow K$, $f \longmapsto (f t_P^{n_P})(P)$ has kernel $L(D-P)$.

$\square$

Rmk For any $f \in \mathcal{K}(C)^\times$ and any divisor $D' \in \mathrm{Div}(C)$, there are only finitely many $D \leq D'$ such that $f \in L(D)$.

Pf $f \in L(D) \iff -\mathrm{div}(f) \leq D \leq D'$
and $D \leq D'$

□

Cor 1.10 For any $D \in \mathrm{Div}(C)$ with $L(D) \neq 0$,

$\ell(D-P) = \ell(D)$ for finitely many $P \in C(\overline{K})$,

and $\ell(D-P) = \ell(D) - 1$ for all other $P \in C(\overline{K})$.

Pf Pick any $0 \neq f \in L(D)$. According to the remark, there are only finitely many P s.t. $f \in L(D-P)$.

□

Lemma 1.11 $\ell(D) + \ell(E) \leq \ell(D+E) + 1 \quad \forall D, E$

Pf Consider the bilinear map

$$L(D) \times L(E) \longrightarrow L(D+E)$$
$$(f, g) \longmapsto fg$$

$$\underbrace{(\mathrm{div}(f) + D)}_{\geq 0} + \underbrace{(\mathrm{div}(g) + E)}_{\geq 0} = \underbrace{\mathrm{div}(fg) + D + E}_{\geq 0}.$$

Let $0 \neq h \in L(D+E)$. There are only fin. many ways of writing $\mathrm{div}(h) + D + E = D' + E'$ with $D', E' \geq 0$. We have $\mathrm{div}(f) + D = \mathrm{div}(f') + D$ if and only if $f' = \lambda f$ for some $\lambda \in K^\times$.

$\Rightarrow$ Any nonempty preimage of any $0 \neq h \in L(D+E)$ has dimension 1.

$\Rightarrow$ $\dim(L(D)) + \dim(L(E)) \leq \dim(L(D+E)) + 1$.

$\square$

1.9. Maps to projective space

Let C be a smooth projective curve.

Def Let $F \subset K(C)$ be an $(n+1)$-dimensional with basis $f_0, \ldots, f_n$. Consider the minimal divisors $D = \sum n_P P$ such that $F \subseteq L(D)$.

$$(-n_P = \min_{f \in F} v_P(f) = \min(v_P(f_0), \ldots, v_P(f_n)).$$

The morphism $\varphi: C \to \mathbb{P}^n_K$ associated to $f_0, \ldots, f_n$ (or to F) is in a neighborhood U of $P \in C(\bar{K})$ given by

$$\varphi(Q) = [(f_0 \cdot t_P^{n_P})(Q) : \quad \cdots \quad : (f_n \cdot t_P^{n_P})(Q)] \text{ for } Q \in U.$$

all well-def. at P,
not all 0 at P
(and therefore in a small nbhd. of P)

Rmk Multiplying $f_0, \ldots, f_n$ by $g \in K(C)^\times$ doesn't change φ.

Rmk This generalizes the earlier construction of the morphism $\varphi: C \to \mathbb{P}^1$ associated to $f \in \mathbb{P}^1_K$ (take $f_0 = f$, $f_1 = 1$).

Thm Every morphism $\varphi: C \to \mathbb{P}^n_K$ whose image isn't contained in a hyperplane in $\mathbb{P}^n_K$ is of this form.

Thm φ is a closed embedding (= isomorphism onto its image) if and only if

$$L(D-P-Q) \subsetneqq L(D-P) \text{ for all } P, Q \in C(\overline{K}).$$

Pf of "$\Rightarrow$" w.l.o.g. $L(D-P) \cap F$ is spanned by $f_1, \dots, f_n$.

$\Rightarrow \varphi(P) = [(f_0 t_P^{n_P})(P) : 0 : \dots : 0] = (1 : 0 : \dots : 0]$

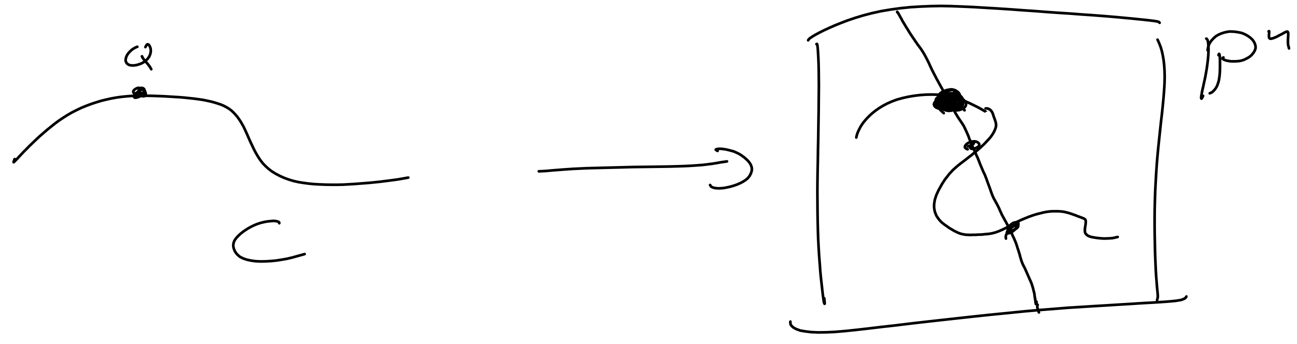
If $L(D-P-Q) = L(D-P)$ for some $Q \neq P$, then $L(D-Q) \subseteq$ $\varphi(Q) = [1 : 0 : \dots : 0]$ for the same reason.

$\Rightarrow \varphi$ isn't injective. $\Rightarrow \varphi$ not an isom. onto its image.

If $L(D-2P) = L(D-P)$, then

$f_1 t_P^{n_P}, \dots, f_n t_P^{n_P}$ have a root of multiplicity at least 2 at P. Hence, the derivative of φ at P is zero. $\Rightarrow \varphi$ not an isom. onto $\varphi(C)$. $\square$

Thm. Let $H \subset \mathbb{P}^n_K$ be a hyperplane which intersects $\varphi(C) \subset \mathbb{P}^n_K$ in the points $P_1, \dots, P_r$ which multiplicities $m_1, \dots, m_r$.
Let $D' = m_1 P_1 + \dots + m_r P_r \in \mathrm{Div}(\varphi(C))$.



Then, $[\varphi^*(D')] = [D]$ in $\mathcal{Cl}(C)$.

Assume φ is a closed embedding.

Sketch of proof W.l.o.g. $H = \{[x_0 : \dots : x_n] \mid x_0 = 0\}$.
Then, $\sum n''_Q Q = D'' := \varphi^*(D') = \mathrm{div}(f_0) + D$:
Let $Q \in C(\overline{K})$ with $\varphi(Q) = P_i$. Then,
$$n''_Q = v_Q(f_0 t_Q^{n_Q}) = v_Q(f_0) + n_Q. \qquad \square$$

1.10. Canonical divisor class

Let C be a sm. proj. curve.

Thm 1.12

a) The module of differentials $\Omega_K(K(C))$ is a one-dimensional $K(C)$-vector space.

b) Let $f \in K(C)$. Then, $df = 0$ if and only if $f \in K$.

Pf b) "$\Leftarrow$" clear

"$\Rightarrow$" Pick $a \in K$ such that $f-a$ has a root $P \in C(\bar{K})$ of mult. 1 (possible since $f: C \to \mathbb{P}^1$ is only ramified at finitely many points).

$\Rightarrow$ f has nonzero derivative at P.

$\Rightarrow df \neq 0$.

a) Let $f, g \in K(C)$. Since $K(C)$ has transcendence degree 1, the elements f, g are algebraically dependent over K.

Let $0 \neq \varphi \in K[S,T]$ be of minimal degree such that $\varphi(f,g) = 0$.

$$\Rightarrow 0 = d\varphi(f,g) = \underbrace{\frac{\partial \varphi}{\partial S}(f,g)}_{\in K(C)} df + \underbrace{\frac{\partial \varphi}{\partial T}(f,g)}_{\in K(C)} dg$$

We can't have $\frac{\partial \varphi}{\partial S}(f,g) = \frac{\partial \varphi}{\partial T}(f,g) = 0$ since $\frac{\partial \varphi}{\partial S} \neq 0$ or $\frac{\partial \varphi}{\partial T} \neq 0$, and both have smaller degree than φ.

$\Rightarrow$ df and dg aren't linearly independent over $K(C)$.

$\square$

Def To a nonzero differential $\omega \in \Omega_K(K(C))$, we associate the divisor

$$\operatorname{div}(\omega) = \sum_{P \in C(\bar{K})} v_P\Big(\underbrace{\frac{\omega}{dt_P}}_{\in K(C) \text{ by Thm 1.12}}\Big) P$$

independent of the choice of uniformizer t_P!

Def The divisors of the form $\operatorname{div}(\omega)$ are called the <u>canonical divisors</u> of C.

Rmk By Thm 1.12, they form a divisor class, denoted by $W = W_C$ (or K_C).

Def The <u>genus</u> of C is $g = g_C = \ell(W) \geq 0$.

Eg $C = \mathbb{P}^1_k$, $\mathcal{O}_{\mathbb{P}^1}(H_0) = K[X^{(0)}_1]$

$$\mathcal{O}_{\mathbb{P}^1}(H_1) = K[X^{(1)}_0]$$

$$X^{(1)}_0 = \left(X^{(0)}_1\right)^{-1}$$

$$\omega = d X^{(0)}_1 = - \frac{d X^{(1)}_0}{(X^{(1)}_0)^2}.$$

$X^{(0)}_1 - a$ is a uniformizer at $a \in \mathbb{A}^1 \cong H_0$.

$X^{(1)}_0 - a$ —"— $a \in \mathbb{A}^1 \cong H_1$.

$$\Rightarrow v_P\left(\frac{\omega}{dt_P}\right) = \begin{cases} 0, & P \neq [0:1], \\ -2, & P = [0:1]. \end{cases}$$

$\Rightarrow \operatorname{div}(\omega) = -2 \cdot [0:1]$

$\Rightarrow W_{\mathbb{P}^1} \in Cl(\mathbb{P}^1) = \mathbb{Z}$ is the divisor class of degree -2.

$\Rightarrow g_{\mathbb{P}^1} = l(-2 \cdot [0:1]) = 0$.