1.8. Divisors

<u>Reference</u> • Fulton, Algebraic Curves, Chapter 8
• Hartshorne, Algebraic Geometry, Chapter IV

Assume char $(K) = 0$.
Let be a smooth projective curve over K.

<u>Def</u> A <u>(Weil) divisor</u> on C (defined over K) is a

formal sum $\sum\limits_{P \in C(\bar{K})} n_P P = \sum\limits_{P \in C(\bar{K})} n_P [P]$

with $n_P \in \mathbb{Z}$ $\forall P$ and $n_P = 0$ for all but finitely many P, which is invariant under the action of $Gal(\bar{K}|K)$: $n_{\sigma(P)} = n_P$ $\forall \sigma \in Gal(\bar{K}|K)$, $P \in C(\bar{K})$.

The (additive) group of divisors is denoted by $Div(C)$.

<u>Equivalent def</u> A Weil divisor is a finite formal sum

$$\sum_{S \subseteq C} n_S S$$

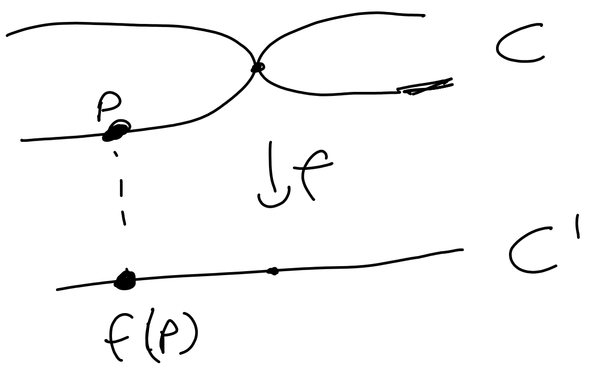
$$\left.\begin{array}{l} S \subseteq C \\ \text{0-dimensional} \\ \text{irreducible} \\ \text{subvarieties} \\ \text{defined over } K \end{array}\right\} = Gal(\bar{K}|K)\text{-orbit of points in } C(\bar{K}).$$

Def The degree of $D = \sum n_P P$ is $\deg(D) = \sum n_P$.

The subgroup of divisors of degree 0 is $\mathrm{Div}^0(C)$.

Def Let $f: C \to C'$ be a morphism between smooth proj. curves over K. The image of $D = \sum n_P P \in \mathrm{Div}(C)$ is

$$f(D) = \sum n_P f(P) \in \mathrm{Div}(C').$$



Rmk $\deg(f(D)) = \deg(D)$.

Def Consider a nonconstant morphism $f: C \to C'$ as above. It induces a field homomorphism

$$\begin{aligned} K(C') &\hookrightarrow K(C) \\ t &\longmapsto t \circ f \end{aligned}$$

$\Rightarrow$ We can interpret $K(C)$ as a field ext. of $K(C')$. The degree of f is

$$\deg(f) := [K(C) : K(C')].$$

For $Q \in C'(\bar{k})$, denote a uniformiser at Q by $t_{C',Q}$. (It's a rational function on C' with coefficients in $\bar{k}$.)

For $P \in C(\bar{k})$, $Q = f(P)$, let

$$e_{P|Q} = v_{C,P}(t_{C',Q} \circ f) \quad (\geq 1),$$

the <u>ramification index</u> of f at P.

<u>Thm</u> For any $Q \in C'(\bar{k})$,

$$\sum_{\substack{P \in C(\bar{k}): \\ f(P) = Q}} e_{P|Q} = \deg(f).$$

Analogy with extensions of number fields

$$\begin{array}{c|c} K(C) & L \\ | & | \\ K(C') & K \end{array}$$

$$\begin{array}{c|c} \mathcal{O}_C(f^{-1}(C' \cap H_i)) & \mathcal{O}_L \\ | & | \\ \mathcal{O}_{C'}(\underbrace{C' \cap H_i}_{\subseteq \mathbb{A}^n}) & \mathcal{O}_K \end{array}$$

$$\begin{array}{c|c} \mathrm{Gal}(\bar{K}|K)\text{-orbits of } P \in f^{-1}(C' \cap H_i)(\bar{K}) & \mathfrak{P} \subseteq \mathcal{O}_L \\ \downarrow f & | \\ \mathrm{Gal}(\bar{K}|K)\text{-orbits of } Q \in (C' \cap H_i)(K') & \mathfrak{q} \subseteq \mathcal{O}_K \end{array}$$

$e_{P|Q}$ | $e_{\mathfrak{P}|\mathfrak{q}}$

$f_{P|Q} = [\underbrace{\mathcal{O}_C(\ldots)/m_P}_{K_P} : \underbrace{\mathcal{O}_{C'}(\ldots)/m_Q}_{K_Q}]$ | $f_{\mathfrak{P}|\mathfrak{q}} = [\mathcal{O}_L/\mathfrak{P} : \mathcal{O}_K/\mathfrak{q}]$

K_P, the smallest field ext. of K s.t. $P \in C(K_P)$

K_Q, the smallest field ext. of K s.t. $Q \in C'(K_Q)$

$$f_{P|Q} = \frac{\text{size of } \mathrm{Gal}(\bar{K}|K)\text{-orbit of } P}{\text{size of } \mathrm{Gal}(\bar{K}|K)\text{-orbit of } Q}$$

$$\sum_{\substack{P \in C(\bar{K}): \\ f(P) = Q}} e_{P|Q} = [K(C) : K(C')] = \deg(f) \quad \Bigg| \quad \sum_{\mathfrak{P}|\mathfrak{q}} e_{\mathfrak{P}|\mathfrak{q}} f_{\mathfrak{P}|\mathfrak{q}} = [L : K]$$

$\mathrm{Div}(C)$ | group of fractional ideals of $\mathcal{O}_L$

<u>Thm</u> $e_{P|Q} = 1$ for all but finitely many points $P \in C(\bar{K})$ $\quad (Q = f(C))$

($\hat{=}\ \mathcal{O}_L | \mathcal{O}_K$ only ramified at finitely many primes)

<u>Def</u> The <u>ramification divisor</u> of f is

$$R_f := \sum_{\substack{P \in C(\bar{K}) \\ Q = f(C)}} (e_{P|Q} - 1) P.$$

($R_f \hat{=}$ different of $\mathcal{O}_L | \mathcal{O}_K$)

($f(R_f) \hat{=}$ discriminant of $\mathcal{O}_L | \mathcal{O}_K$).

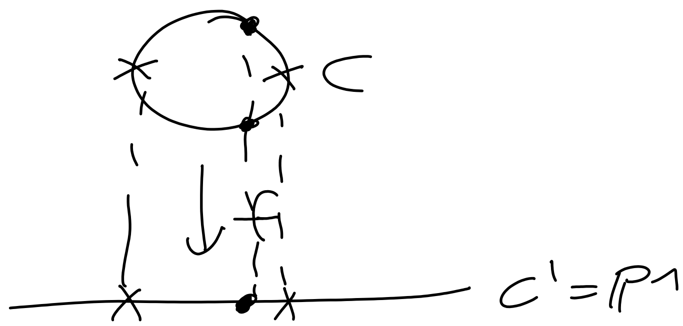
Es. $C = \{[x:y:z] \mid x^2+y^2 = z^2\} \subset \mathbb{P}^2$

$$\downarrow f$$

$C' = \mathbb{P}^1$

$$C \xrightarrow{f} C'$$

$$[x:y:z] \longmapsto [x:z]$$



Look at the restriction

$$\begin{array}{ccc} C \cap H_2 & \xrightarrow{f} & C' \cap H_1 \\ \| & & \| \\ \{(x,y) \in \mathbb{A}^2 \mid x^2+y^2=1\} & & \mathbb{A}^1 = \{s \in \mathbb{A}^1\} \end{array}$$

$$K(C') = K(C' \cap H_1) = K(s)$$

$$\downarrow$$

$$K(C) = K(C \cap H_2) = K(x)[y]/(x^2+y^2-1)$$

$$K(C') \longrightarrow K(C)$$

$$s \longmapsto x$$

$$\deg(f) = [K(x)[y]/(x^2+y^2-1) : K(x)] = 2$$

The preimages of $s \in \mathbb{A}^1$ are $(s, \pm\sqrt{1-s^2}) \in C(\overline{k})$.

~~If~~ If $s \neq \pm 1$, there are two preimages, each with multiplicity 1.

If $s = \pm 1$, there is one preimage, with multiplicity 2.

Also check the point $\infty = [1:0] \in \mathbb{P}^1$: There are two preimages $[1 : \pm\sqrt{-1} : 0]$, each with multiplicity 1.

$$\Rightarrow R_f = \underset{\mathbb{A}^2 \subset \mathbb{P}^2}{(1,0)} + (-1,0) = [1:0:1] + [-1:0:1].$$

Def The preimage of $D' = \sum n_Q Q \in \mathrm{Div}(C')$ is
$$f^*(D') = \sum_{\substack{P \in C(\bar{K}) \\ f(P) = Q}} n_Q e_{P|Q} P.$$

Cor a) $f(f^*(D')) = \deg(f) \cdot D'$

b) $\deg(f^*(D')) = \deg(f) \cdot \deg(D')$

Def To a rational function $f \in K(C)^\times$, we associate the divisor

$$\mathrm{div}(f) = \sum_{P \in C(\overline{K})} \underbrace{v_{C,P}(f)}_{\substack{>0 \text{ iff } f \text{ has a zero at } P \\ <0 \text{ iff } f \text{ has a pole at } P}} P$$

Rmk $\mathrm{div}: K(C)^\times \longrightarrow \mathrm{Div}(C)$ is group hom.

Def The divisor class group of C is

$$\mathrm{Cl}(C) := \mathrm{Div}(C) / K(C)^\times$$ (the cokernel of the map $\mathrm{div}: K(C)^\times \to \mathrm{Div}(C)$).

($\hat{=}$ ideal class group)

Thm $\deg(\mathrm{div}(f)) = 0 \quad \forall f \in K(C)^\times$

(Number of zeros with mult. = number of poles with mult.)

Pf If $f \neq 0$ is constant, $\mathrm{div}(f) = 0$.

If f is nonconstant, interpret it as $f: C \to \mathbb{P}^1$.

$$\Rightarrow \mathrm{div}(f) = f^*([0] - [\infty])$$

$$[0], [\infty] \in \mathbb{P}^1$$

$$\Rightarrow \deg(\mathrm{div}(f)) = \deg(f) \cdot \underbrace{\deg([0] - [\infty])}_{0} = 0 \qquad \square$$

Thm $\operatorname{div}(f) = 0 \iff f = \text{constant}$

Pf If $f \neq$ const, then $f : C(\bar{K}) \to \mathbb{P}^1(\bar{K})$ is surjective. $\Rightarrow f$ has a zero. $\Rightarrow \operatorname{div}(f) \neq 0$ $\square$

Cor $\mathcal{O}_C(C) = K$.

Pf $f : C(\bar{K}) \to \mathbb{P}^1(\bar{K})$ surjective
$\Rightarrow f$ has a pole ($=$ preimage of ∞) $\square$

Def $\mathcal{Cl}^0(C) := \operatorname{Div}^0(C)/K(C)^\times$.

Rmk The image of $\deg : \operatorname{Div}(C) \to \mathbb{Z}$ is nonzero (take $D = \operatorname{Gal}(\bar{K}|K)$-orbit of any point $P \in C(\bar{K})$).

$\Rightarrow \deg(\operatorname{Div}(C)) \cong \mathbb{Z}$

$\Rightarrow \operatorname{Div}(C) \cong \operatorname{Div}^0(C) \times \mathbb{Z}$

$\mathcal{Cl}(C) \cong \mathcal{Cl}^0(C) \times \mathbb{Z}$

Warning The map $\deg : \operatorname{Div}(C) \to \mathbb{Z}$ might not be surjective.

Ese $\deg : Cl(\mathbb{P}1) \longrightarrow \mathbb{Z}$ is an isomorphism.

Pf surjective: $\deg([O]) = 1$.

injective: Let $D = \sum_P n_P P \in Div^0(C)$.

Take $f(X,Y) = \prod_{a \in \overline{K} \subset \mathbb{P}1} (X - aY)^{n_{[a]}} \cdot X^{n_{[\infty]}}$.

Since $\sum n_P = 0$, the numerator and denominator of f are homogeneous of the same degree, so $f(X,Y) \in K(\mathbb{P}1)$.

Furthermore,

$$\begin{aligned} div(f) &= \sum_{a \in \overline{K}} n_{[a]}[a] + n_{[\infty]}[\infty] \\ &= \sum n_P P = D. \end{aligned}$$

$\square$

Def We write $D \leq D'$ if $n_P \leq n_P'$ $\forall P$.
$$\underset{\sum n_P P}{\overset{\shortparallel}{D}} \leq \underset{\sum n_P' P}{\overset{\shortparallel}{D'}}$$

D is <u>effective</u> if $D \geq 0$.

Def For any $D \in \mathrm{Div}(C)$, we let

$$L(D) = \{ f \in K(C)^\times \mid \mathrm{div}(f) + D \geq 0 \} \cup \{0\}.$$

Lemma $L(D)$ is a K-vector space.

Pf • $\mathrm{div}(\lambda f) = \mathrm{div}(f)$ $\forall \lambda \in K^\times$

• $v_P(f+g) \geq \min(v_P(f), v_P(g))$

nonarch.
triangle inequality

"if f has a root of order a at P,
and g — " — b at P,
then $f+g$ — " — $\geq \min(a,b)$ at P"

$\square$

Def $\ell(D) := \dim(L(D))$ as a K-vector space.