Def $\emptyset \neq V \subseteq \mathbb{P}^n_K$ is irreducible if we can't write $V = V_1 \cup V_2$ with projective varieties $V_1, V_2 \subsetneq V$ defined over K.

Rmk If $\emptyset \neq V \subseteq \mathbb{P}^n_K$ is irreducible, then for all i, $V \cap H_i = \emptyset$ or $\varphi_i(V \cap H_i) \subseteq \mathbb{A}^n_K$ is irreducible (as a variety in $\mathbb{A}^n_K$).

Warning The converse doesn't hold!

E.g. $V = \{[1:0], [0:1]\} \subset \mathbb{P}^1_K$ isn't irreducible, but $V \cap H_0 = \{[1:0]\}$, $V \cap H_1 = \{[0:1]\}$ are!

Reminder We've covered $\mathbb{P}^n_K$ by open subsets $H_0, \dots, H_n$ and defined isomorphisms $\varphi_i : H_i \to \mathbb{A}^n_K$.
(bijections, homeomorphisms)

$$\mathbb{P}^n_K \supseteq H_i \xrightarrow[\varphi_i]{\sim} \mathbb{A}^n_K$$

$$[x_0 : \dots : x_n] \longmapsto \left(x_j^{(i)}\right)_{j \neq i},$$

(with $x_i \neq 0$)

where $x_j^{(i)} = \dfrac{x_j}{x_i}$

$$\left(x_i^{(i)} = \frac{x_i}{x_i} = 1 \right)$$

Change of coordinates:

$$\begin{tikzcd}[column sep=small] & {[x_0:\cdots:x_n] \in H_i \cap H_j} \arrow[dl, mapsto, "\varphi_i"'] \arrow[dr, mapsto, "\varphi_j"] & \\ \mathbb{A}^n_k \ni (x^{(i)}_k)_{k\neq i} \arrow[rr, "\psi_{ij}"] & & (x^{(j)}_k)_{k\neq j} \in \mathbb{A}^n_k, \end{tikzcd}$$

where $\boxed{x^{(j)}_k = \frac{x_k}{x_j} = \frac{\frac{x_k}{x_i}}{\frac{x_j}{x_i}} = \frac{x^{(i)}_k}{x^{(i)}_j}}$

This defines an isomorphism (bij, homeon.) between the open subset

$\varphi_i(H_i \cap H_j) = \{(x^{(i)}_k)_{k\neq i} \mid x^{(i)}_j \neq 0\}$ of $\mathbb{A}^n_k$

and the open subset

$\varphi_j(H_i \cap H_j) = \{(x^{(j)}_k)_{k\neq j} \mid x^{(j)}_i \neq 0\}$ of $\mathbb{A}^n_k$

We can then define a function on $V \subseteq \mathbb{P}^n_k$ (or on an open subset U of V) to be a collection of functions $f_0, \dots, f_n$ on $\underset{\mathbb{A}^n_k}{\underset{\cap}{\varphi_0(V \cap H_0)}}, \dots, \underset{\mathbb{A}^n_k}{\underset{\cap}{\varphi_n(V \cap H_n)}}$ (def. on $\varphi_0(U \cap H_0), \dots$) so that f_i and f_j agree on $V \cap H_i \cap H_j$ (on $U \cap H_i \cap H_j$).

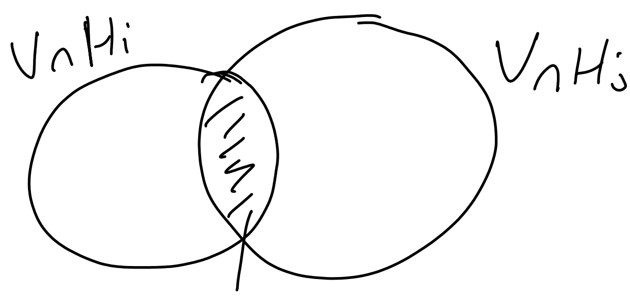
$$\rightsquigarrow \mathcal{O}_V(U) = \Big\{ (f_0, \dots, f_n) \in \prod_i \mathcal{O}_{\varphi_i(V \cap H_i)}(\varphi_i(U \cap H_i)) \;\Big|\; f_i|_{U \cap H_i \cap H_j} = f_j|_{U \cap H_i \cap H_j} \; \forall i,j \Big\}.$$

Def For any irreducible $V \subseteq \mathbb{P}^n_k$, its field of rational functions is

$$K(V) = K(\underbrace{\varphi_i(V \cap H_i)}_{\subseteq \mathbb{A}^n_k})$$ for any i such that $V \cap H_i \neq \emptyset$.

Rmk This is independent of i: since V is irreducible, we have

$V \cap H_i \cap H_j \neq \emptyset$ whenever $V \cap H_i \neq \emptyset$ and $V \cap H_j \neq \emptyset$



$V \cap H_i \cap H_j \neq \emptyset$

Rmk $$K(V) = \bigcup_{\emptyset \neq U \subseteq V \text{ open}} \mathcal{O}_V(U)$$

Prob $\mathcal{O}_{\mathbb{P}^n_K}(\mathbb{P}^n_K) = K$, the ring of constant functions.

Pf Elements of $\mathcal{O}_{\mathbb{P}^n_K}(\mathbb{P}^n_K)$ correspond to tuples $(f_0, \dots, f_n)$, where $f_i \in \mathcal{O}_{\mathbb{A}^n_K}(\mathbb{A}^n_K)$ and $f_j = f_i \circ \varphi_{ij}$.

f_j is a polynomial in the variables

$$x_k^{(j)} = \frac{x_k^{(i)}}{x_j^{(i)}}.$$

But if f_j is a nonconstant polynomial in these variables, then f_i cannot be a polynomial in the variables $x_k^{(i)}$ (for any $i \neq j$).

$\Rightarrow f_i \notin \mathcal{O}_{\mathbb{A}^n_K}(\mathbb{A}^n_K)$. ↯ $\square$

Ese The "function" $\mathbb{P}^1_K \longrightarrow K$ has a

$$[x:y] \longmapsto \frac{x}{y}$$

pole at $[0:1]$.

Summary

$K(V)$ = field of rat. fcts

$\mathcal{O}_V(V)$ = ring of fcts. on V defined everywhere on V

$\mathcal{O}_V(U)$ = ring of fcts. on V defined everywhere on U

$\mathcal{O}_{V,p}$ = ring of fcts. on V defined at p

Rmk $K(\mathbb{P}^n_k) = \left\{ \frac{f}{g} \;\middle|\; f, g \in k[x_0, \dots, x_n] \text{ homogenous of the same degreed, } g \neq 0 \right\}$

Note: $$\frac{f(\lambda x_0, \dots, \lambda x_n)}{g(\lambda x_0, \dots, \lambda x_n)} = \frac{\cancel{\lambda^d} f(x_0, \dots, x_n)}{\cancel{\lambda^d} g(x_0, \dots, x_n)},$$

so $\frac{f}{g}(x)$ is independent of the choice of representative $(x_0, \dots, x_n)$ of $[x_0 : \dots : x_n] \in \mathbb{P}^n_k$.

Def For a k-vector space $L \subseteq k[x_0, \dots, x_n]$ and $d \geq 0$, let $L_d \subseteq L$ be the subspace of homogeneous degree d polynomials.

Rmk Let $V \subseteq \mathbb{P}^n_K$ be irreducible.

Then, $K(V) = \left\{ \frac{f}{g} \;\middle|\; \begin{array}{l} f, g \in K[x_0, \dots, x_n]_d / I(V)_d \\ \text{for some } d \geq 0 \\ g \neq 0 \end{array} \right\}$

Def If $V \subseteq \mathbb{P}^n_K$ is irreducible,

$\dim(V) :=$ transcendence degree of $K(V)$

$= \dim(\varphi_i(V \cap H_i))$ if $V \cap H_i \neq \emptyset$.

Ex $\dim(\mathbb{P}^n_K) = n$.

Def An irred. variety $V \subseteq \mathbb{P}^n_K$ is smooth if

$\varphi_i(V \cap H_i) \subseteq \mathbb{A}^n_K$ is smooth for all i such that $V \cap H_i$.

The tangent space at $P \in V(K)$ is

$T_{V,P} := T_{\underbrace{\varphi_i(V \cap H_i)}_{\subseteq \mathbb{A}^n_K}, \varphi_i(P)}$ for any i such that $P \in H_i(K)$.

$\mathcal{O}_{V,P} := \mathcal{O}_{\varphi_i(V \cap H_i), \varphi_i(P)}$ for any i such that $P \in H_i(K)$.

Def Let $V \subseteq \mathbb{P}^n_K$ and $W \subseteq \mathbb{P}^m_K$.

A morphism $f: V \longrightarrow W$ is a map

$$f: V(\overline{K}) \longrightarrow W(\overline{K}) \text{ which satisfies}$$

one of the following equivalent conditions:

a) There is a covering of V by (finitely many) open sets U_s ($s \in S$) such that for each s, there are functions

$$f_0, \dots, f_m \in \mathcal{O}_V(U_s) \text{ such that}$$

$$f(P) = [f_0(P) : \dots : f_m(P)] \quad \forall P \in U_s(\overline{K}).$$

b) $f: V(\overline{K}) \to W(\overline{K})$ is continuous w.r.t. the Zariski topologies over K and for $i = 0, \dots, m$, there are functions $f_j^{(i)} \in \mathcal{O}_V(f^{-1}(H_i))$ $(f_i^{(i)} = 1)$ such that

$$\underbrace{\varphi_i(\underbrace{f(P)}_{\in H_i})}_{\in \mathbb{A}^n_K} = \underbrace{(f_0^{(i)}, \dots, \widehat{f_i^{(i)}}, \dots, f_n^{(i)})}_{\in \mathbb{A}^n_K} \quad \forall P \in f^{-1}(H_i).$$

Exe $\mathbb{P}^1_k \longrightarrow \mathbb{P}^{d+1}_k$ (<u>Veronese embedding of degree d</u>)

$$[x:y] \longmapsto [x^d : x^{d-1}y : \dots : xy^{d-1} : y^d]$$

<u>not</u> rational functions on $\mathbb{P}^1_k$.

$$= \left[\left(\frac{x}{y}\right)^d : \left(\frac{x}{y}\right)^{d-1} : \dots : 1\right]$$

fct. on $\mathbb{P}^1_k$ defined on $\{[x:y] \mid y \neq 0\}$

$$= \left[1 : \dots : \left(\frac{y}{x}\right)^{d-1} : \left(\frac{y}{x}\right)^d\right]$$

fcts on $\mathbb{P}^1_k$ defined on $\{[x:y] \mid x \neq 0\}$

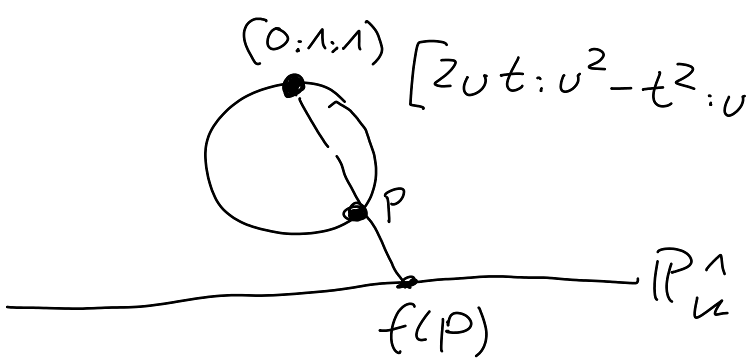
$\{y \neq 0\}$ and $\{x \neq 0\}$ form an open cover of $\mathbb{P}^1_k$.

Exe $\mathbb{P}^2_k \supseteq \{[x:y:z] \mid x^2+y^2=z^2\} =: C \longrightarrow \mathbb{P}^1_K$ (see first lecture)

$[x:y:z] \longmapsto [x : y-z]$ if $x \neq 0$ or $y-z \neq 0$

$= [y+z : -x]$ if $y+z \neq 0$ or $-x \neq 0$

$[2ut : u^2-t^2 : u^2+t^2] \longleftarrow\!\shortmid [t:u]$



Ese Embedding $\mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k$ $(n \leq m)$

$$[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_n : 0 : \cdots : 0]$$

Warning There is no "projection" morphism

$$f : \mathbb{P}^2_k \longrightarrow \mathbb{P}^1_k$$
$$[x : y : z] \longmapsto [x : y] \text{ for } (x, y) \neq (0,0)$$

Pf $f([0 : y : 1]) = [0 : y] = [0 : 1]$ $\forall y \neq 0$

$f([x : 0 : 1]) = [x : 0] = [1 : 0]$ $\forall x \neq 0$

$\Rightarrow$ By continuity, $f([0 : 0 : 1]) = [0 : 1]$

and $f([0 : 0 : 1]) = [1 : 0]$. $\lightning$ $\square$

Lemma Let C be a smooth projective curve and let $t \in k(C)$. Then, there is a morphism $C \longrightarrow \mathbb{P}^1_k$

$$P \longmapsto \begin{cases} \underbrace{[t(P) : 1]}_{\simeq t(P)} & \text{if } t \text{ is defined at } P, \\ \underbrace{[0 : 1]}_{\simeq \infty} & \text{if } t \text{ isn't defined at } P \end{cases}$$

(="pole at P")

Pf $k(C)$ is the field of fractions of $\mathcal{O}_{V,P}$.

t defined at $P \iff v_{V,P}(t) \geq 0$

$\frac{1}{t}$ defined at $P \iff v_{V,P}(t) \leq 0$

Here, we use smoothness!

We can define

$$C \longrightarrow \mathbb{P}^1_k$$

$$P \longmapsto \begin{cases} [t(P):1] & \text{if } t \text{ is def. at } P, \\ [1:\frac{1}{t}(P)] & \text{if } \frac{1}{t} \text{ is def. at } P. \end{cases}$$

$\square$

Rmk The lemma fails for singular curves! (It's possible that P looks like a zero when approaching in one direction and like a pole in a different direction).

Reference

Hartshorne, Algebraic Geometry, Chapter I.