Math 137: Algebraic Geometry

Spring 2021

Problem set #9

due Friday, April 16 at noon

On this problem set, K is any field (not necessarily algebraically closed).

Any invertible linear map $g : K^{n+1} \to K^{n+1}$ induces a map $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ sending the line spanned by $x \in K^{n+1}$ to the line spanned by $g(x) \in K^{n+1}$. Maps $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ of this form are called *projective transformations.*

Problem 1. a) Consider the projective line $\mathbb{P}^1_K = K \sqcup \{\infty\}$. Let P, Q, R be three distinct points in $\mathbb{P}^1_K$. Show that there is a projective transformation $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ sending P to 0, Q to 1, and R to ∞.

b) We say that points $P_1, \ldots, P_m$ in $\mathbb{P}^n_K$ are *in general linear position* if no $d + 2$ of them lie on a d-dimensional linear subspace for any $0 \leqslant d \leqslant \min(m - 2, n - 1)$.

Let $P_1, \ldots, P_{n+2} \in \mathbb{P}^n_K$ be in general linear position and let $Q_1, \ldots, Q_{n+2} \in \mathbb{P}^n_K$ be in general linear position. Show that there is a unique projective transformation $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ sending P_i to Q_i for $i = 1, \ldots, n + 2$.

Problem 2. Assume that K is algebraically closed. Show that a polynomial $f \in K[X_1, \ldots, X_n]$ vanishes on the entire line spanned by a nonzero vector $x \in K^n$ if and only if all of its homogeneous parts f_d vanish at x.

Problem 3. Consider a finite field $\mathbb{F}_q$ of size q.

a) How many points are there in $\mathbb{P}^n_{\mathbb{F}_q}$?

b) For $0 \leqslant d \leqslant n$, how many d-dimensional linear subspaces does $\mathbb{P}^n_{\mathbb{F}_q}$ have?

c) For $0 \leqslant d' \leqslant d \leqslant n$ and a d'-dimensional linear subspace L of $\mathbb{P}^n_{\mathbb{F}_q}$, how many d-dimensional linear subspaces M containing L does $\mathbb{P}^n_{\mathbb{F}_q}$ have?

Problem 4. Let $A = V(I)$ for an ideal I of $K[X_1, \ldots, X_n]$. Let $S \subseteq K[X_0, \ldots, X_n]$ be the set of homogenizations of elements of I at X_0. Show that $V_{\mathbb{P}^n_K}(S)$ is the Zariski closure of the image of A under the 0-th standard affine chart map φ_0.

Problem 5 (Pappus's hexagon theorem)**.** Let $g \neq h$ be lines in $\mathbb{P}^2_K$ that intersect in the point P. Let A, B, C be points on g and A', B', C' be points on h (all seven points P, A, B, C, A', B', C' distinct). Let Z be the point of intersection of the lines AB' and $A'B$. Let Y be the point of intersection of the lines AC' and $A'C$. Let X be the point of intersection of the lines BC' and $B'C$. Show that X, Y, Z are colinear. (Hint: Apply a projective transformation to for example make $P = [0 : 0 : 1]$, $A = [1 : 0 : 0]$, $B = [1 : 0 : 1]$, $C = [r : 0 : 1]$, $A' = [0 : 1 : 1]$, $B' = [0 : 1 : 0]$, $C' = [0 : s : 1]$. Then compute X, Y, Z.)