Math 137: Algebraic Geometry

Spring 2021

Problem set #8

due Monday, April 5 at noon

(You can still submit solutions to the double bonus problem 3b from the previous problem set.)

Problem 1. We call $P \in K^n$ a *point of symmetry* for a subset $S \subseteq K^n$ if the reflection $2P - Q$ across P of any point $Q \in S$ lies in S. Show that any nonempty algebraic subset $S \subseteq K^n$ that doesn't contain a straight line has at most one point of symmetry.

Problem 2. Let $\varphi : V \to W$ be a dominant morphism between irreducible algebraic sets. Assume that there is a nonempty Zariski open subset U of W such that $|\varphi(w)| < \infty$ for all $w \in U$. Show that $\dim(V) = \dim(W)$.

Problem 3. For $r \leqslant n$, consider the set $V_r \subseteq M_n(K)$ of $n \times n$-matrices of rank at most r. You've shown on problem set 3 that V_r is an algebraic subset of $M_n(K) = K^{n\times n}$. Show that its dimension is $2nr - r^2$.

Problem 4. Let $V \subseteq K^n$ be an irreducible algebraic set and let $P \in K^n$ be a point not contained in V. Show that the Zariski closure of the join of V and $\{P\}$ has dimension $\dim(V) + 1$.

Problem 5 (bonus)**.** Let $n \geqslant 3$ and $1 \leqslant d \leqslant 2n-4$. Let $f \in K[X_1, \dots, X_n]$ be a polynomial of degree at most d. Show that if $V(f) \subseteq K^n$ contains a straight line, then it contains infinitely many straight lines.

Problem 6 (bonus)**.**

a) Let $V \subseteq K^n$ be an algebraic subset of dimension m. Let $P_d \subset K[X_1, \dots, X_n]$ be the vector space of polynomials of degree at most d and let $Q_d \subseteq \Gamma(V)$ be the set of restrictions of elements of P_d to V. Show that there is a number C such that for all $d \geqslant 1$, we have $\dim_K(Q_d) \leqslant C \cdot d^m$. (Hint: Use a linear projection arising from Noether normalization)

b) Let $V_1, \dots, V_m$ be any irreducible algebraic subsets of K^n of codimension at least 2. Show that there is an irreducible algebraic subset $W \subsetneq K^n$ containing $V_1 \cup \dots \cup V_m$.