Math 137: Algebraic Geometry

Spring 2021

Problem set #10

due Friday, April 23 at noon

On this problem set, K is an algebraically closed field.

Problem 1. Let A and B be algebraic subsets of K^n and let $P \in A \cap B$.

a) Show that the tangent cone of $A \cup B$ at P is the union of the tangent cones of A at P and of B at P.

b) Is the tangent cone of $A \cap B$ at P always the intersection of the tangent cones of A at P and of B at P?

Problem 2. Let $A \subseteq K^n$ and $B \subseteq K^m$ be algebraic and let $\varphi : A \to B$ be an isomorphism.

a) Show that the tangent spaces of A at P and of B at $f(P)$ are isomorphic for all $P \in A$. (Hint: Look at the derivative of f at P.)

b) (bonus) Show that the tangent cones of A at P and of B at $f(P)$ are isomorphic for all $P \in A$.

Problem 3. Consider one-dimensional vector spaces $V_1, \ldots, V_n \subset K^2$ and points $P_1, \ldots, P_n \in K^2$. Show that there is a one-dimensional algebraic subset $C \subset K^2$ containing $P_1, \ldots, P_n$ and such that V_i is the tangent space of C at P_i for $i = 1, \ldots, n$. (Bonus points if you show that there is an irreducible such set C.)

Problem 4. Consider the (algebraic) set $\mathrm{SL}_n(K) \subset K^{n^2}$ of $n \times n$-matrices M with $\det(M) = 1$.

a) Show that $\mathrm{SL}_n(K)$ is irreducible.

b) What is the tangent space of $\mathrm{SL}_n(K)$ at the identity matrix $I_n \in SL_n(K)$?

c) Show that $\mathrm{SL}_n(K)$ is smooth. (Hint: Make use of the maps $\varphi_N : \mathrm{SL}_n(K) \to \mathrm{SL}_n(K)$ sending M to NM for $N \in \mathrm{SL}_n(K)$.)

Problem 5 (bonus, Bertini's theorem)**.** Let $V \subseteq K^n$ be an (irreducible) smooth algebraic set. Show that there is an affine hyperplane $H \subset K^n$ such that every irreducible component of $V \cap H$ is smooth. (Hint: dimension counting)