Math 137: Algebraic Geometry

Spring 2021

Problem set #1

due Monday, February 8 at noon

Problem 1. Let K be a field and let X be a set of m points in K^n.

a) Show that there is a set $S \subseteq K[X_1, \ldots, X_n]$ of size at most n^m such that $X = V(S)$.

b) Assuming that $K = \mathbb{R}$, show that there is a polynomial $f \in K[X_1, \ldots, X_n]$ such that $X = V(f)$.

c) (bonus) Assuming that the field K is finite, show that there is a polynomial $f \in K[X_1, \ldots, X_n]$ such that $X = V(f)$. (Hint: Use Fermat's little theorem / Lagrange's theorem.)

d) (bonus) Assuming that the field K is infinite, show that there is a set $S \subseteq K[X_1, \ldots, X_n]$ of size at most $n + 1$ such that $X = V(S)$.

Problem 2. Show that $X = \{(t, \sin(t)) \mid t \in \mathbb{R}\}$ is not an algebraic subset of $\mathbb{R}^2$.

Problem 3. Consider the *one-sheet hyperboloid*

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 + 1\} \subseteq \mathbb{R}^3.$$

Prove that every point $P \in V$ lies on exactly two (straight) lines $l_1, l_2 \subseteq V$.

Problem 4. For every $n \geqslant 1$, show that the ideal $I = (X, Y)^n$ of $K[X, Y]$ is not generated by n of its elements.

Problem 5. a) Let A be an algebraic subset of K^n and let B be an algebraic subset of K^m. Show that the cartesian product $A \times B$ is an algebraic subset of $K^n \times K^m = K^{n+m}$.

b) Let $K = \mathbb{R}$. Show that the Zariski topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is **not** the product topology arising from the Zariski topology on $\mathbb{R}$. (In other words, show that there is a Zariski closed subset of $\mathbb{R}^2$ that is not the intersection of sets of the form $A \times B$, where A and B are Zariski closed subsets of $\mathbb{R}$.)