Pf of Thm 2.75 Let $R = K[b_1, \dots, b_m]$.

$\Rightarrow L = K(b_1, \dots, b_m)$. $\Rightarrow m \geq n$

Induction over m.

$m = n$: done!

$m > n$: $\Rightarrow b_1, \dots, b_m$ are algebraically dependent over K.

Let $0 \neq f \in K[X_1, \dots, X_m]$ with $f(b_1, \dots, b_m) = 0$.

If $f(b_1, \dots, b_{m-1}, X) \in K[b_1, \dots, b_{m-1}][X]$ is monic, then b_m is integral over $K[b_1, \dots, b_{m-1}]$ and we can proceed by induction.

Let $d \geq 1$ be the degree of $f(X_1, \dots, X_m)$.

Consider the polynomial

$$g(X) := f(b_1 + c_1(X - b_m), \dots, b_{m-1} + c_{m-1}(X - b_m), X)$$
$$= f(b_1 - c_1 b_m + c_1 X, \dots, b_{m-1} - c_{m-1} b_m + c_{m-1} X, X)$$
$$\in K[b_1 - c_1 b_m, \dots, b_{m-1} - c_{m-1} b_m][X]$$

for $c_1, \dots, c_{m-1} \in K$.

Note that $g(b_m) = f(b_1, \dots, b_{m-1}, b_m) = 0$.

The degree of $g(x)$ is at most d and its

X^d-coefficient is some <u>nonzero</u> polynomial $h(c_1,\dots,c_{m-1})$ in $c_1,\dots,c_{m-1}$ with $h\in K[Y_1,\dots,Y_{m-1}]$. (Just expand!)

By the Nichtnullstellensatz, there exist $c_1,\dots,c_{m-1}\in K$ such that $l := h(c_1,\dots,c_{m-1})\neq 0$.

Then, $\frac{1}{l}\cdot g(X)$ is a monic polynomial with coeff. in $K[b_1-c_1b_m,\dots,b_{m-1}-c_{m-1}b_m]$ which is zero for $X=b_m$.

$\Rightarrow b_m$ is integral over

$$K[b_1-c_1b_m,\dots,b_{m-1}-c_{m-1}b_m].$$

$\Rightarrow$ We can proceed by induction.

$\square$

<u>Nagata's trick</u> In the proof, one could instead use

$$g(X)=f(b_1+(X-b_m)^{d_1},\dots,b_{m-1}+(X-b_m)^{d_{m-1}},X)$$

for appropriate $d_1,\dots,d_{m-1}\geq 0$.

2.17. Another definition of dimension

<u>Lemma 2.76</u> Let $V \subsetneq W$ be irreducible alg. sets. Then, $\dim(V) \leq \dim(W)-1$ and there is an irreducible alg. set $V \subseteq A \subsetneq W$ of dimension $\dim(A) = \dim(W)-1$.

<u>Rmk</u> It's important that W is irreducible! (otherwise, take $V = \{P\}$, $W = \{P, Q\}$.)

<u>Pf</u> Let $n = \dim(W)$. By Noether Normalization, there is a dominant finite morphism $\varphi : W \to K^n$.

$$V \subsetneq W \underset{\text{incomparability}}{\Longrightarrow} \varphi(V) \underset{\text{closed}}{\subsetneq} \varphi(W) = K^n$$

Take $0 \neq f \in K[X_1, \dots, X_n]$ which vanishes on $\varphi(V)$: $\varphi(V) \subseteq V(f)$.

Since $\varphi(V)$ is irreducible, it is contained in some irreducible component of $V(f)$, which corresponds to some irreducible factor of f.

$\Rightarrow$ We can assume that f is irreducible.

$$\dim(V) \underset{\text{Thm 2.68}}{=} \dim(\varphi(V)) \leq \dim(V(f)) = n-1.$$

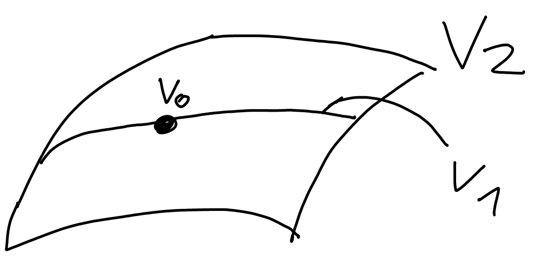
Since $V \subseteq \varphi^{-1}(V(f))$ is irreducible, it is contained in some irreducible component A of $\varphi^{-1}(V(f)) = V(\varphi^*(f))$.

$$\underset{\substack{\text{Thm 2.74} \\ \text{(Going down)}}}{\Longrightarrow} \quad \varphi(A) = V(f) \Rightarrow \dim(A) = \dim(\varphi(A)) = \dim(V(f)) = n-1.$$

$\square$

<u>Cor 2.77</u> Let $V \neq \emptyset$ be any algebraic set. Then, $\dim(V)$ is the largest $d \geq 0$ such that there are irreducible alg. sets

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d \subseteq V.$$



<u>Pf</u> <u>$\dim(V) \geq d$:</u>

$0 \leq \dim(V_0) < \dim(V_1) < \cdots < \dim(V_d) \leq \dim(V)$

<u>$\dim(V) \leq d$:</u> W.l.o.g. V is irreducible. Take $V_d = V$.

If $\dim(V) \geq 1$, then $\{P\} \subsetneq V$ for any $P \in V$.

$\Rightarrow \exists\ \{P\} \subseteq V_{d-1} \subsetneq V_d$ of dimension $\dim(V)-1$.

Continue the chain (induction ...) $\square$

Cor 2.78 Let $V \subseteq W$ both be irreducible.

Then, the <u>codimension</u>

$\operatorname{codim}(V, W) := \dim(W) - \dim(V)$

of ~~V~~ in W is the largest $c \geq 0$ such that there are irreducible alg. sets

$V = V_0 \subsetneq \cdots \subsetneq V_c = W$.

Pf "as before" $\square$

2.18. Defining with few equations

Def An irreducible $(n-1)$-dimensional irreducible subset of K^n is called a <u>hypersurface</u> in K^n.

Thm 2.79 Any hypersurface $V \subseteq K^n$ is of the form $V = V(f)$ for some irreducible $0 \neq f \in K[X_1, \dots, X_n]$.

Pf Since $V \subsetneq K^n$, there exists some irreducible $f \neq 0$ s.t. $\underbrace{V}_{\text{irred.}} \subseteq \underbrace{V(f)}_{\text{irred.}}$.

If $V \subsetneq V(f)$, then $\dim(V) < \dim(V(f)) = n-1$. $\lightning$

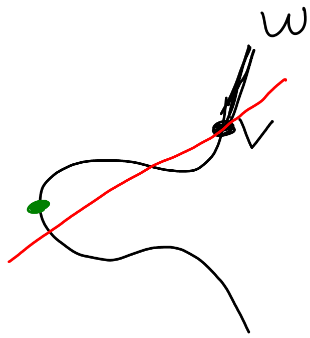
$\Rightarrow V = V(f)$. $\square$

Cor 2.80 Let $V \subseteq W$ both be irreducible. Then, there are $c := \operatorname{codim}(V, W)$ functions $f_1, \dots, f_c \in \Gamma(W)$ such that V is an irreducible component of $V(f_1, \dots, f_c)$ $= \{ p \in W \mid f_1(p) = \dots = f_c(p) = 0 \} \subseteq W$ and all other irreducible components also have codimension c in W.

$\rightsquigarrow$ "Essentially, c functions suffice to define an irreducible subset of codimension c."

Rmk Let $K = \mathbb{C}$,

$W = \{(x,y) \mid y^2 = x^3 - 4x + 4\}$ (irred.)



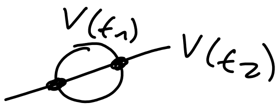
$V = \{(2,2)\}$ or $\{(\pi, \sqrt{\pi^3 - 4\pi + 4})\}$.

There is no function $f \in \Gamma(W)$ such that $V = V(f)$.

Pf Difficult!

Rmk Assuming $W = K^n$, I don't know whether there are always functions $f_1, \dots, f_c \in K[X_1, \dots, X_n]$ such that $V = V(f_1, \dots, f_c)$!

(Problem: Even if $f_1, \dots, f_c$ are irreducible, $V(f_1, \dots, f_c)$ might not be!



)

Pf of Cor 2.80

Induction over $c = \mathrm{codim}(V, W)$.

Let $c \geq 1$.

Let $\varphi : W \longrightarrow K^n$ be a dominant finite morphism with $n = \dim(W)$.

$\Rightarrow \mathrm{codim}(\varphi(V), K^n) = \mathrm{codim}(V, W) = c$.

Let $0 \neq g_1 \in K[X_1, \dots, X_n]$ be irreducible

$\varphi(V) \subseteq V(g_1) \subsetneq K^n$.

$\Rightarrow \mathrm{codim}(\varphi(V), V(g_1)) = c - 1$.

By induction, there are $g_2, \dots, g_c \in K[X_1, \dots, X_n]$ such that $\varphi(V)$ is an irred. comp. of $V(g_1, \dots, g_c)$ and all other irred. comp. also have codimension c in K^n. The preimage $\varphi^{-1}(V(g_1, \dots, g_c)) = V(\underbrace{\varphi^*(g_1)}_{f_1}, \dots, \underbrace{\varphi^*(g_c)}_{f_c})$ has irreducible components of codimension c.

$V \subseteq \varphi^{-1}(V(g_1, \dots, g_c))$ is irreducible and hence contained in some irred. comp. Since dimensions match, V is actually equal to this component!

$\square$