# Math 286X: Arithmetic Statistics

## Spring 2020

### Problem set #5

**Problem 1.** Consider a measure-preserving action of a countable group $G$ on a set $X$. Let $\widetilde{\mathcal{F}}$ be a measurable almost fundamental domain for this action, with volume $0 < \mathrm{vol}(\widetilde{\mathcal{F}}) < \infty$. Show that the corresponding fundamental domain $\mathcal{F}$ also has volume $0 < \mathrm{vol}(\mathcal{F}) < \infty$.

**Problem 2.** Order the full integer lattices $\Lambda \subseteq \mathbb{Z}^n$ by their covolume.

a) Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$. Show that

$$\mathbb{P}(e_1 \in \Lambda \mid \Lambda \subseteq \mathbb{Z}^n \text{ full lattice}) = 0.$$

b) Let $\pi : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ be the projection onto the first $n-1$ coordinates. Then, $\pi(\Lambda) \subseteq \mathbb{Z}^{n-1}$ is always a full lattice. Show that

$$\mathbb{P}(\pi(\Lambda) = \mathbb{Z}^{n-1} \mid \Lambda \subseteq \mathbb{Z}^n \text{ full lattice}) = \frac{1}{\zeta(2)\cdots\zeta(n)}.$$

**Problem 3** (Mahler's criterion)**.** Equip $\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$ with the quotient topology. Let $X$ be a closed subset of $\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$. Show that $X$ is compact if and only if there exist $0 < C \leqslant C' < \infty$ such that the successive minima $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ of any lattice $\Lambda$ corresponding to an element of $X$ satisfy $C \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant C'$.
**Hint:** Use the Iwasawa decomposition and Siegel's almost fundamental domain.

**Problem 4** (lattice points in cusps)**.** For any $\alpha \in \mathbb{R}$ and any $X > 0$, consider the compact set

$$S_\alpha(X) = \{(x,y) \in \mathbb{R}^2 \mid 1 \leqslant x \leqslant X \text{ and } |y - \alpha x| \leqslant \frac{1}{x}\}$$

and let $N_\alpha(X) = \#(S_\alpha(X) \cap \mathbb{Z}^2)$.

a) Show that its Lebesgue measure is $\mathrm{vol}(S_\alpha(X)) = 2\log X$.

b) (strangeness) Let $\alpha = \frac{p}{q}$ with $\gcd(p, q) = 1$. Show that

$$N_\alpha(X) = \frac{X}{q} + \mathcal{O}(q).$$

c) (Dirichlet's approximation theorem) Show that for any $\alpha \in \mathbb{R}$, we have

$$\lim_{X\to\infty} N_\alpha(X) = \infty.$$

d) (averaging) For any $a < b$, show that

$$\frac{1}{b-a}\int_a^b N_\alpha(X)\mathrm{d}\alpha = 2\log X + \mathcal{O}\left(1 + \frac{1}{b-a}\right).$$

**Problem 5** (smooth functions are swell)**.** Let $A$ be a weighted set on $\mathbb{R}$ whose characteristic function $\chi_A : \mathbb{R} \to \mathbb{R}^{\geqslant 0}$ is smooth and whose support is bounded. Let $k \geqslant 0$. Show that

$$\#((T \cdot A) \cap \mathbb{Z}) = T \cdot \mathrm{vol}(A) + \mathcal{O}_{A,k}(T^{-k})$$

for $T \to \infty$. (Note that the error term is much better than the error term $\mathcal{O}(1)$ we would get if $A$ were an interval!)
**Hint:** For example, apply the Poisson summation formula or the Euler–Maclaurin formula (both use integration by parts).

**Problem 6.** Fix a number $n \geqslant 1$ and let $\mathrm{GL}_n(\mathbb{R}) = NAK$ and $\mathrm{SL}_n(\mathbb{R}) = NA_1K_1$ be the Iwasawa decompositions defined in class.

a) Let $B = (\mathbb{R}^{>0})^{n-1}$ (with Haar measure $\mathrm{d}^\times\mathfrak{b} = \prod_i \mathfrak{b}_i^{-1}\mathrm{d}\mathfrak{b}_i$) and consider the isomorphism $B \to A_1$ sending $\mathfrak{b} \in B$ to $\mathfrak{a} \in A_1$, where $\mathfrak{b}_i^n = \mathfrak{a}_{i+1}/\mathfrak{a}_i$ and conversely $\mathfrak{a}_i = (\mathfrak{b}_1 \cdots \mathfrak{b}_{i-1})^n/(\mathfrak{b}_1^{n-1} \cdots \mathfrak{b}_{n-1})$. Show that the pull-back of the Haar measure on $\mathrm{SL}_n(\mathbb{R})$ defined in class along the diffeomorphism $N \times B \times K_1 \to \mathrm{SL}_n(\mathbb{R})$ arising from the Iwasawa decomposition is as follows (with $(\mathfrak{n}, \mathfrak{b}, \mathfrak{k}) \in N \times B \times K_1$):

$$n^{n-2} \cdot \prod_{j=1}^{n-1} \mathfrak{b}_j^{-nj(n-j)}\mathrm{d}^\times\mathfrak{n}\mathrm{d}^\times\mathfrak{b}\mathrm{d}^\times\mathfrak{k}.$$

b) Let $\widetilde{\mathcal{F}} = N'A'K \subset \mathrm{GL}_n(\mathbb{R})$ be Siegel's almost fundamental domain for $\mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{R})$. Compute the volume of $\widetilde{\mathcal{F}} \cap \mathrm{SL}_n(\mathbb{R})$ with respect to the Haar measure on $\mathrm{SL}_n(\mathbb{R})$ defined in class.

**Problem 7.** Don't worry, be happy.