Math 286X: Arithmetic Statistics

Spring 2020

Problem set #4

Problem 1 (Compare with problem 2 on problem set 3)**.** Let $A \subset \mathbb{R}$ be a compact subset and let $I \subset \mathbb{R}$ be a bounded interval. Let $B \subset \mathbb{R}$ be the weighted set whose characteristic function is the convolution

$$\chi_B(x) = \frac{1}{\mathrm{vol}(I)} \cdot \int_{\mathbb{R}} \chi_A(x-s)\chi_I(s)\mathrm{d}s = \frac{1}{\mathrm{vol}(I)} \cdot \int_{\mathbb{R}} \chi_A(s)\chi_I(x-s)\mathrm{d}s.$$

Show that

$$\#((T \cdot B) \cap \mathbb{Z}) \sim_{A,I} \mathrm{vol}(A) \cdot T$$

for $T \to \infty$.

Problem 2. Explicitly describe fundamental domains for the following actions:

a) The action of $\mathbb{Z}_p$ on $\mathbb{Q}_p$ by translation.

b) The action of $\mathbb{Z}_{(p)} = \{p^a b \mid a, b \in \mathbb{Z}\} \subset \mathbb{Q}$ on $\mathbb{R} \times \mathbb{Q}_p$ given by $g.(x, y) = (g + x, g + y)$.

Problem 3. Let $\mathcal{V}(\mathbb{Z})$ be the set of quadratic forms $aX^2 + bXY + cY^2$ with $a, b, c \in \mathbb{Z}$, ordered by $\max(|a|, |b|, |c|)$. Let p be a prime number. Call an integer $D \in \mathbb{Z}$ *fundamental at* p if $p^2 \nmid D$ when $p \neq 2$ and if $D \equiv 1 \mod 4$ or $D \equiv 8, 12 \mod 16$ when $p = 2$. (This means that $D \neq 1$ is a fundamental discriminant if and only if it is fundamental at every prime p.) Show that

$$\mathbb{P}(\mathrm{disc}(f) \text{ is fundamental at } p \mid f \in \mathcal{V}(\mathbb{Z})) = 1 - p^{-2} - p^{-3} + p^{-4}.$$

(Feel free to use a computer.)

Problem 4. Let K be a quadratic number field of discriminant D. In class, we've constructed a bijection

$$\mathrm{Cl}_K = K^\times \backslash \{I \text{ fractional ideal of } K\} \longleftrightarrow \mathrm{GL}_2(\mathbb{Z}) \backslash \mathcal{V}_{\mathrm{disc}=D}(\mathbb{Z}).$$

Let $\mathcal{W}(\mathbb{Z}) = \mathcal{V}(\mathbb{Z}) \times \mathbb{Z}^2$ be the set of pairs $e = (f, v)$, where f is a binary quadratic form with integer coefficients, and $v \in \mathbb{Z}^2$. Let $\mathrm{disc}(e) = \mathrm{disc}(f)$

and $\mathrm{Nm}(e) = f(v)$. Furthermore, let $\mathrm{GL}_2(\mathbb{Z})$ act on $\mathcal{W}(\mathbb{Z})$ by $M.(f, v) = (M.f, \det(M)(M^T)^{-1}v)$ (where the action on $\mathcal{V}(\mathbb{Z})$ was defined in class by $(M.f)(w) = f(M^T w)/\det(M)$). For any $N \geqslant 1$, let $\mathcal{W}_{\mathrm{disc}=D, |\mathrm{Nm}|=N} \subset \mathcal{W}$ be the set of $e \in \mathcal{W}$ with $\mathrm{disc}(e) = D$ and $|\mathrm{Nm}(e)| = N$.

a) Construct a bijection

$$\{I \subseteq \mathcal{O}_K \text{ ideal of } \mathcal{O}_K \mid \mathrm{Nm}(I) = N\} \longleftrightarrow \mathrm{GL}_2(\mathbb{Z})\backslash\mathcal{W}_{\mathrm{disc}=D, |\mathrm{Nm}|=N}(\mathbb{Z}).$$

b) What is the $\mathrm{GL}_2(\mathbb{Z})$-stabilizer of an element of $\mathcal{W}_{\mathrm{disc}=D, |\mathrm{Nm}|=N}(\mathbb{Z})$?