GETTING AWAY WITHOUT INFRASTRUCTURES

1. A generalization of fundamental domains

Let G be a topological group acting transitively on a set X. Fix some $x_0 \in X$ and assume that the stabilizer $S_0 = \mathrm{Stab}_G(x_0)$ is a unimodular group, with Haar measure $\mathrm{d}s$. Write any $x \in X$ as $x = gx_0$ with $g \in G$. Of course, $\mathrm{Stab}_G(x) = gS_0g^{-1}$ is isomorphic to S_0, so we obtain a Haar measure on $\mathrm{Stab}_G(x)$ for any $x \in X$. (Note that it is independent of the choice of g.)

Let H be a countable subgroup of G and assume that $\mathcal{F}$ is a fundamental domain for the left action of H on G. We then define a weighted set $\mathcal{A}$ on X by letting

$$\chi_{\mathcal{A}}(x) = \int_{S_0} \chi_{\mathcal{F}}(gs)\mathrm{d}s = \int_{\mathrm{Stab}_G(x)} \chi_{\mathcal{F}}(sg)\mathrm{d}s$$

for $x = gx_0 \in X$. (Assuming the integral exists!) This is again independent of the choice of g.

The following lemma shows that $\mathcal{A}$ is some sort of generalization of a fundamental domain for the action of H on X, which allows us not to count orbits, but to find the sum of the covolume of the orbits' H-stabilizers in the G-stabilizers.

Lemma 1. *Summing over the elements of an H-orbit Hx_1 in X, we have*

$$\sum_{x \in Hx_1} \chi_{\mathcal{A}}(x) = \mathrm{vol}(\mathrm{Stab}_H(x_1)\backslash \mathrm{Stab}_G(x_1)),$$

the volume of a measurable fundamental domain $\mathcal{B}$ for the left action of $\mathrm{Stab}_H(x_1)$ on $\mathrm{Stab}_G(x_1)$.

Proof. Let $x_1 = gx_0$. We then have

$$\begin{aligned}
\sum_{x\in Hx_1} \chi_{\mathcal{A}}(x) &= \sum_{h\in H/\operatorname{Stab}_H(x_1)} \chi_{\mathcal{A}}(hx_1) \\
&= \sum_{h\in H/\operatorname{Stab}_H(x_1)} \int_{\operatorname{Stab}_G(hx_1)} \chi_{\mathcal{F}}(shg)\mathrm{d}s \\
&= \sum_{h\in H/\operatorname{Stab}_H(x_1)} \int_{\operatorname{Stab}_G(x_1)} \chi_{\mathcal{F}}(hsg)\mathrm{d}s \\
&= \sum_{h\in H/\operatorname{Stab}_H(x_1)} \int_{\operatorname{Stab}_G(x_1)} \sum_{t\in\operatorname{Stab}_H(x_1)} \chi_{\mathcal{B}}(ts)\chi_{\mathcal{F}}(hsg)\mathrm{d}s \\
&= \sum_{h\in H/\operatorname{Stab}_H(x_1)} \int_{\operatorname{Stab}_G(x_1)} \sum_{t\in\operatorname{Stab}_H(x_1)} \chi_{\mathcal{B}}(s)\chi_{\mathcal{F}}(ht^{-1}sg)\mathrm{d}s \\
&= \sum_{h\in H/\operatorname{Stab}_H(x_1)} \int_{\operatorname{Stab}_G(x_1)} \sum_{t\in\operatorname{Stab}_H(x_1)} \chi_{\mathcal{B}}(s)\chi_{\mathcal{F}}(htsg)\mathrm{d}s \\
&= \int_{\operatorname{Stab}_G(x_1)} \chi_{\mathcal{B}}(s) \sum_{h\in H} \chi_{\mathcal{F}}(hsg)\mathrm{d}s \\
&= \int_{\operatorname{Stab}_G(x_1)} \chi_{\mathcal{B}}(s)\mathrm{d}s \\
&= \operatorname{vol}(\mathcal{B}) \\
&= \operatorname{vol}(\operatorname{Stab}_H(x_1)\backslash \operatorname{Stab}_G(x_1)).
\end{aligned}$$

□

2. Application to ideal class group of quadratic number fields

Fix a fundamental discriminant D and let $K = \mathbb{Q}(\sqrt{D})$. In class, we constructed a $\mathrm{GL}_2(\mathbb{Z})$-equivariant bijection

$$K^\times\backslash\{\mathbb{Z}\text{-bases } (\omega_1,\omega_2) \text{ of fractional ideals of } K\} \longleftrightarrow \mathcal{V}_{\mathrm{disc}=D}(\mathbb{Z}).$$

Using the same formula, we can extend it to a $\mathrm{GL}_2(\mathbb{R})$-equivariant bijection

$$(K\otimes\mathbb{R})^\times\backslash\{\mathbb{R}\text{-bases } (\omega_1,\omega_2) \text{ of } K\otimes\mathbb{R}\} \longleftrightarrow \mathcal{V}_{\mathrm{disc}=D}(\mathbb{R}).$$

The first bijection proves that for any $f \in \mathcal{V}_{\mathrm{disc}=D}(\mathbb{Z})$ corresponding to a fractional ideal $I = \langle\omega_1,\omega_2\rangle$, we have

$$\operatorname{Stab}_{\mathrm{GL}_2(\mathbb{Z})}(f) \cong \{\alpha\in K^\times \mid \alpha I = I\} = \mathcal{O}_K^\times.$$

The second bijection proves that for any $f \in \mathcal{V}_{\mathrm{disc}=D}(\mathbb{R})$ we have

$$\operatorname{Stab}_{\mathrm{GL}_2(\mathbb{R})}(f) \cong (K\otimes\mathbb{R})^\times.$$

Letting $\mathrm{GL}_2^{\pm1}(\mathbb{R})$ be the set of $M \in \mathrm{GL}_2(\mathbb{R})$ with $\det(M) = \pm1$, we get

$$\operatorname{Stab}_{\mathrm{GL}_2^{\pm1}(\mathbb{R})}(f) \cong (K\otimes\mathbb{R})^{\mathrm{Nm}=\pm1},$$

the set of $\alpha \in (K\otimes\mathbb{R})^\times$ with $\mathrm{Nm}_{K\otimes\mathbb{R}|\mathbb{R}}(\alpha) = \pm1$.

If $D < 0$, we have $K \otimes \mathbb{R} \cong \mathbb{C}$, so this is the group S^1 of complex numbers on the unit circle. For example, endow this group with the "standard" Haar measure such that $\mathrm{vol}(S^1) = 2\pi$. Then,

$$\mathrm{vol}(\mathrm{Stab}_{\mathrm{GL}_2(\mathbb{Z})}(f) \backslash \mathrm{Stab}_{\mathrm{GL}_2^{\pm 1}(\mathbb{R})}(f)) = \mathrm{vol}(\mathcal{O}_K^\times \backslash (K \otimes \mathbb{R})^{\mathrm{Nm}=\pm 1}) = \frac{2\pi}{\#\mathcal{O}_K^\times}.$$

If $D > 0$, we have $K \otimes \mathbb{R} \cong \mathbb{R} \times \mathbb{R}$, so this is the group of $(\alpha_1, \alpha_2) \in \mathbb{R}^\times \times \mathbb{R}^\times$ with $\alpha_1\alpha_2 = \pm 1$ (isomorphic to $\mathbb{R}^\times \times \{\pm 1\}$). Endowing this group with the "standard" Haar measure (product of the standard Haar measure $\mathrm{d}x/|x|$ on $\mathbb{R}^\times$ and the counting measure on $\{\pm 1\}$), we get

$$\mathrm{vol}(\mathrm{Stab}_{\mathrm{GL}_2(\mathbb{Z})}(f) \backslash \mathrm{Stab}_{\mathrm{GL}_2^{\pm 1}(\mathbb{R})}(f)) = \mathrm{vol}(\mathcal{O}_K^\times \backslash (K \otimes \mathbb{R})^{\mathrm{Nm}=\pm 1}) = R_K,$$

the regulator of K.

We apply section 1 with $X = \mathcal{V}_{\mathrm{disc}=D}(\mathbb{R})$, $G = \mathrm{GL}_2^{\pm 1}(\mathbb{R})$, $H = \mathrm{GL}_2(\mathbb{Z})$. For x_0, take for example the quadratic form $\frac{\sqrt{-D}}{2} \cdot (X^2 + Y^2)$ if $D < 0$ and the quadratic form $\sqrt{D} \cdot XY$ if $D > 0$. For $\mathcal{F}$, take for example the restriction to $\mathrm{GL}_2^{\pm 1}(\mathbb{R})$ of Minkowski's fundamental for the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathrm{GL}_2(\mathbb{R})$ if $D < 0$, and its transpose inverse for $D > 0$ (makes the computation simpler).

Then, Lemma 1 tells you that summing $\chi_{\mathcal{A}}(f)$ over elements f of $\mathcal{V}_{\mathrm{disc}=D}(\mathbb{Z})$ gives you the sum over all ideal classes of K of $2\pi/\#\mathcal{O}_K^\times$ if $D < 0$ and of R_K if $D > 0$. And the characteristic function $\chi_{\mathcal{A}}$ is sufficiently nice that you can estimate this sum using a volume computation.