p-adic change of variables

(see Igusa: An introduction to the theory of local zeta functions, pg. 111
León-Cardenal, Zúñiga-Galindo: ... from scratch)

<u>Thm</u> (change of var. in dim. 1) Let K be a nonarch. local field and let $U \subset K$ be a compact open subset and $f(X) \in K[X]$. For any $y \in K$, let $m(y)$ be be the number of $x \in U$ s.t. $f(x) = y$. Then,

ASIDE
$$\underbrace{\int_K m(y)\,dy}_{\mathrm{vol}(\mathrm{im}(f: U \to K) \text{ as a multiset})} = \int_U |f'(x)|\,dx$$

<u>Exe</u> Let $K = \mathbb{Q}_p$, $U = \mathbb{Z}_p^\times$, $f(x) = X^2$.

If $p \neq 2$: By Hensel's lemma, for $y \in \mathbb{Z}_p$,

$$m(y) = \begin{cases} 2, & (y \bmod p) \in \mathbb{F}_p^{\times 2} \text{ (quadr. res.)} \\ 0, & \text{else.} \end{cases}$$

$$\Rightarrow LHS = 2 \cdot \frac{\#\text{nonzero quadr. res.}}{p} = \frac{p-1}{p} = 1 - \frac{1}{p}$$

$$v_p(f'(x)) = v_p(2x) = 0 \quad \forall x \in \mathbb{Z}_p^\times \Rightarrow |f'(x)| = 1 \; \forall x \in \mathbb{Z}_p^\times$$

$$\Rightarrow RHS = \int_{\mathbb{Z}_p^\times} 1\,dx = \mathrm{vol}(\mathbb{Z}_p^\times) = 1 - \frac{1}{p} \quad \checkmark$$

If $p=2$: By Hensel's lemma, for $y \in \mathbb{Z}_2^\times$:

$$m(y) = \begin{cases} 2, & y \equiv 1 \bmod 8 \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow LHS = 2 \cdot \frac{1}{8} = \frac{1}{4}$$

$v_2(f'(x)) = v_2(2x) = 1$, so $|f'(x)| = \frac{1}{2}$ $\forall x \in \mathbb{Z}_2^\times$

$$\Rightarrow RHS = \int_{\mathbb{Z}_2^\times} \frac{1}{2}\, dx = \frac{1}{2} \operatorname{vol}(\mathbb{Z}_2^\times) = \frac{1}{4} \quad \checkmark$$

Ex Let $K = \mathbb{F}_p((T))$, $U = \underset{\mathcal{O}_K}{\overset{\parallel}{\mathbb{F}_p[[T]]}}$, $f(x) = X^p$.

For $y \in \mathbb{F}_p[[T]]$:

$$m(y) = \begin{cases} 1, & y = b_0 + b_p X^p + b_{2p} X^{2p} + \dots \text{ for some } b_0, b_p, \dots \in \mathbb{F}_p \\ 0, & \text{else} \end{cases}$$

(∞ many digits have to be 0)

$$\Rightarrow LHS = 0$$

$$|f'(x)| = |pX^{p-1}| = 0$$

$$\Rightarrow RHS = 0 \quad \checkmark$$

Pf of Thm. Replace U by $\pi^a U$ and $\pi^b f\left(\frac{X}{\pi^a}\right)$.

$\Rightarrow$ we can assume that $U \subseteq \mathcal{O}_K$ and $f(X) \in \mathcal{O}_K[X]$.

The map $\begin{array}{l} U \longrightarrow \mathbb{Z} \cup \{\infty\} \\ x \longmapsto v(f'(x)) \end{array}$ is continuous.

You can show that $\mathrm{vol}(f(\{x \in \mathcal{O}_K \mid f'(x) = 0\})) = 0$.

(If the pol. $f'(X)$ is nonzero, it's a finite set.

Otherwise, $f(X)$ is constant or $\mathrm{char}(K) = p > 0$ and $f(X) = g(X^p)$ for some pol. $g(X) \in \mathcal{O}_K(X)$.

$$\begin{array}{ccccc} \mathcal{O}_K & \longrightarrow & \mathcal{O}_K & \longrightarrow & \mathcal{O}_K \\ x \mapsto x^p & & x \longmapsto f(x) & & \end{array}$$

By the last ex. the image of $x \mapsto x^p$ has volume 0.

$\Rightarrow$ the image of $x \mapsto g(x^p)$ has volume 0.)

The sets $\{x \in U \mid v(f'(x)) = t\}$ for $t \in \mathbb{Z}$ are also compact and open. $\rightsquigarrow$ w.l.o.g. $v(f'(x)) = t \;\forall x \in U$.

For large enough e, we have $a + \pi^e \subseteq U \;\forall a \in U$ (because U is compact and open) and

$f(a + \pi^e) = f(a) + \pi^{t+e}$ and each $y \in f(a) + \pi^{t+e}$ has exactly one preimage in $a + \pi^e$ (by Hensel's lemma). We have

$$\int_{a+\pi^e} \underbrace{|f'(x)|}_{q^{-t}} \, dx = q^{-e-t} = \int_{f(a)+\pi^{e+t}} 1 \, dy.$$

$\Rightarrow$ The result follows by splitting up U into sets of the form $a+p^e$ for $a \in U$. $\square$

More generally:

<u>Thm</u> Let $U \subset K^n$ be a cpt. open set and $f_1(x), \ldots, f_n(x) \in K[X_1, \ldots, X_n]$. For any $y \in K^n$, let $m(y)$ be the number of $x \in K^n$ s.t. $f(x) = y$.

Then,
$$\int_K m(y)\,dy = \int_U |\det \mathrm{Jac}(f)(x)|\,dx,$$
where $\mathrm{Jac}(f)(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right)_{i,j}$.

<u>Pf</u> "as in the real case". $\square$

Fixing an $\mathcal{O}_K$-basis $(\omega_1, \dots, \omega_n)$ of $\mathcal{O}_L$, we can identify $\mathcal{O}_L$ with $\mathcal{O}_K^n$.

$$b_1\omega_1 + \dots \longleftrightarrow (b_1, \dots, b_n)$$

The Haar measures on $\mathcal{O}_L$ and $\mathcal{O}_K^n$ agree.

The map

$$\begin{array}{ccc} \varphi : \mathcal{O}_L & \longrightarrow & \mathcal{O}_K^n \\ \wr\| & & \\ \mathcal{O}_K^n & & \\ (b_1, \dots, b_n) & \longmapsto & \prod_{i=1}^{n} (X - \sigma_i(b_1\omega_1 + \dots + b_n\omega_n)) \end{array}$$

sending $\alpha \in \mathcal{O}_L$ to its min. pol. is given by n polynomials in $b_1, \dots, b_n$.

<u>Claim</u> The Jacobian det. at $\pi \in U_L \subseteq \mathcal{O}_L \cong \mathcal{O}_K^n$ is $|D_{L|K}|$.

$$\underset{\substack{\uparrow \\ \text{change of var.}}}{\Longrightarrow} \ \mathrm{vol}(\varphi(U_L) \text{ as a multiset}) = \mathrm{vol}(U_L) \cdot |D_{L|K}|$$

$$= q^{-1}(1 - q^{-1}) \cdot |D_{L|K}|.$$

Since $\varphi : \bigsqcup U_L \longrightarrow P_n$ is an n-cover,

$$\sum_{L \subseteq K^{sep}} \mathrm{vol}(\varphi(U_L) \text{ as multiset}) = n \cdot \mathrm{vol}(P_n)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\sum_L q^{-1}(1 - q^{-1}) \cdot |D_{L|K}| \qquad\qquad n \cdot q^{-(n-1)} \cdot q^{-1}(1 - q^{-1})$$

$$\Longrightarrow \frac{1}{n} \sum_L |D_{L|K}| = \frac{1}{q^{n-1}}.$$

$\square$

Pf of claim W.l.o.g., the basis of $\mathcal{O}_L$ is given

$w_i = \pi^{i-1}$ $(i = 1, \dots, n)$. The map φ is the

composition of

$$\mathcal{O}_K^n \cong \mathcal{O}_L \longrightarrow \mathcal{O}_L^n$$
$$\alpha \longmapsto (\sigma_j(\alpha))_j$$
$$(b_1, \dots, b_n) \longmapsto \Big(\sum_i b_i \, \sigma_j(\pi^{i-1}) \Big)_j$$

and $\mathcal{O}_L^n \longrightarrow \mathcal{O}_K^n$.

$$(c_j)_j \longmapsto \prod_j (x - c_j)$$

The first map has Jacobian matrix $\left(\sigma_j(\pi^{i-1}) \right)_{i,j}$.

at π.

The second map has Jacobian determinant at $(\sigma_j(\pi))_j$

$$\pm \prod_{i<j} (\sigma_i(\pi) - \sigma_j(\pi)) = \pm \det\left((\sigma_j(\pi^{i-1}))_{i,j} \right)$$

by problem 3a on PSet 3.

$\Rightarrow$ The absolute Jacobian det. of φ at π is

$$\left| \det(\sigma_j(\pi^{i-1}))_{i,j} \right|^2 = |D_{L|K}| . \qquad \square$$

$(\pi^{i-1})_i$ is a basis of $\mathcal{O}_L$ over $\mathcal{O}_K$

Thm Let K be a nonarch. local field. Consider the (separable) deg. n field ext. $L|K$ with ram. index e and res. field ext. deg. f $(n = e \cdot f)$. We have

$$\frac{1}{n} \sum_{L \subseteq K^{sep}} |D_{L|K}| = \sum_{\substack{L \text{ up} \\ \text{to isom.}}} \frac{|D_{L|K}|}{\# \mathrm{Aut}_K(L)} = \frac{1}{f \cdot q^{n-f}} .$$

Pf

$$\begin{tikzcd}
L \arrow[d, dash, "\text{deg. } e \text{ tot. ram.}"] \\
L^{I(L|K)} = F \arrow[d, dash, "\text{deg. } f \text{ unram.}"] \\
K
\end{tikzcd}$$

There is exactly one unram. deg. f ext. $F|K$.

By the rel. disc. formula,

$$D_{L|K} = Nm_{F|K}(D_{L|F}) \cdot \underbrace{D_{F|K}}_{\substack{(1) \text{ because} \\ F|K \text{ is unram.}}} = Nm_{F|K}(D_{L|F})$$

$$\Rightarrow |D_{L|K}|_K = |Nm_{F|K}(D_{L|F})|_K = |D_{L|F}|_F$$

$$\Rightarrow \frac{1}{n} \sum_{L \subseteq K \text{ sep.}} |D_{L|K}|_K$$

$$= \frac{1}{n} \sum |D_{L|F}|_F$$

$$= \frac{1}{f \cdot e} \sum |D_{L|F}|_F$$

$$= \frac{1}{f} \cdot \frac{1}{(q^f)^{e-1}} = \frac{1}{f q^{n-f}}. \qquad \square$$

res. field of F is $\mathbb{F}_{q^f}$

Thm Let K be a nonarch. local field. Consider the nondeg. deg. n ext. $L|K$. We have

$$\sum_{\substack{L \text{ up to} \\ \text{isom.}}} \frac{|D_{L|K}|}{\# \mathrm{Aut}_K(L)} = \sum_{k=0}^{n} \frac{P(n,k)}{f^{n-k}},$$

where $P(n,k)$ is the number of partitions of the integer n into k positive summands (modulo order).