Reminder: $M_n^T(\mathbb{Z}) = \{ g \in M_n^+(\mathbb{Z}) \mid 0 < \det(g) \le T \}$ (AS, 98)

$\Rightarrow$ It remains to prove:

**Lemma** $\#\left(SL_n(\mathbb{Z}) \backslash M_n^T(\mathbb{Z})\right) \sim \frac{1}{n} \zeta(2) \cdots \zeta(n) \cdot T^n$ for $T \to \infty$.

There's a better fund. dom. for [the action on integral matrices] $SL_n(\mathbb{Z}) \backslash M_n^+(\mathbb{Z})$:

Any $SL_n(\mathbb{Z})$-orbit contains exactly one matrix $g \in M_n^+(\mathbb{Z})$ of the form
$$g = \begin{pmatrix} a_1 & b_{12} & \cdots & b_{1n} \\ & a_2 & \ddots & b_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_n \end{pmatrix} \quad \text{with } a_1, \ldots, a_n \ge 1$$
and $0 \le b_{ij} < a_j$ for all $i < j$. (Hermite normal form)

[Construct it column by column, from left to right. In the $i$-th column, first use the Euclidean algorithm to make rows $i, \ldots, n$ look right. ($a_i$ is the gcd of the original entries in these $n-i+1$ places.) Then subtract/add row $i$ from/to rows $1, \ldots, i-1$ to make them correct.]

$$\Rightarrow \#\left(SL_n(\mathbb{Z}) \backslash M_n^T(\mathbb{Z})\right) = \sum_{\substack{a_1, \ldots, a_n \ge 1 \\ a_1 \cdots a_n \le T}} \underbrace{a_2^1 a_3^2 \cdots a_n^{n-1}}_{\substack{\text{number of} \\ \text{possible values of } b_{ij}}}$$

The Dirichlet series of $c_k := \sum_{a_1 \cdots a_n = k} a_1^0 a_2^1 \cdots a_n^{n-1}$ is $\zeta(s) \zeta(s-1) \cdots \zeta(s-n+1)$.
Its rightmost pole is at $s = n$, of order 1, with residue $\zeta(n) \cdots \zeta(2)$.

$$\Rightarrow \underset{\substack{\uparrow \\ \text{Wiener-Ikehara}}}{\sum_{k \le T} c_k} \underset{\substack{\uparrow \\ \text{HW?}}}{\sim} \frac{1}{n} \zeta(2) \cdots \zeta(n) \cdot T^n \quad \text{for } T \to \infty.$$

$\square$

[We still need to show that the number of int. matrices in a (measurable) fund. dom. $\mathcal{F}$ is asymptotic to vol($\mathcal{F}$). Problem: counting int. pts. in measurable sets can go horribly wrong!]

(AS, 99)

## Convolution

[Don't panic. Convolve!]

**Def** Let $G$ be a unimodular group with Haar measure $dg$. The convolution of two measurable sets $A, B$ on $G$ is the set $A*B$ with char. fct.

$$\chi_{A*B}(g) = \int_G \chi_A(s)\chi_B(s^{-1}g)\,ds \quad \left[= \int_A \chi_{sB}(g)\,ds\right]$$

$$= \int_G \chi_A(gt^{-1})\chi_B(t)\,dt \quad \left[= \int_B \chi_{At}(g)\,dt\right]$$

$t = s^{-1}g$

Haar measure is inv. under right mult. by $g$ and under inversion by unimodularity

Shorthand: $\chi_{A*B} = \int_A \chi_{\{sB\}}\,ds = \int_B \chi_{\{At\}}\,dt$.

**Rmk** ~~$A*B$ is well-defined if and only if~~

$A*B$ well-defined

$\Leftrightarrow \int_G \chi_A(s)\chi_B(s^{-1}g)\,ds < \infty$ for all $g \in G$

$\Leftrightarrow \int_G \chi_A(gt^{-1})\chi_B(t)\,dt < \infty$ for all $g \in G$

~~Exe If $\chi_A$ is bounded (e.g. if $A$ is a set) and vol$(B)<\infty$, then $A*B$ is well-defined.~~

~~Rmk Since $\chi_B(s^{-1}g) = \chi_{sB}(g)$ and $\chi_A(gt^{-1}) = \chi_{At}(g)$, it's reasonable to write $A*B = \int_A (sB)\,ds = \int_B (At)\,dt$.~~

**Exe** If the char. fct. $\chi_A$ is bounded (e.g. if $A$ is a set) and vol$(B)<\infty$, then $A*B$ is well-defined.

**Rmk** $A*B$ is measurable and

$$\mathrm{vol}(A*B)=\mathrm{vol}(A)\cdot\mathrm{vol}(B)$$

**Pf** $\int_G \chi_{A*B}(g)\,dg = \int_G\int_G \chi_A(s)\chi_B(s^{-1}g)\,ds\,dg = \int_G \chi_A(s)\underbrace{\int_G \chi_B(s^{-1}g)\,dg}_{\mathrm{vol}(B)}\,ds = \mathrm{vol}(A)\cdot\mathrm{vol}(B)$. □

**Rmk** $B*A = (A^{-1}*B^{-1})^{-1}$ ~~In general not commutative!~~

**Rmk** If $G$ is commutative, then $B*A=A*B$.

**Rmk** $A*(B*C)=(A*B)*C$

**Idea**

a) horrible $*$ nice $=$ nice, where "nice" means e.g. "smooth" or "easy to count lattice points in"

"Convolving with an interval fills in small holes."

[See problem 2 on PSet 3 and problem 1 on PSet 4.]

"It also thickens cusps, making them easier to understand."

b) (fund. dom.) $*$ (set of volume 1) $=$ (fund. dom.)

[The combination of these two facts is very powerful!]

**Thm** Let $A, B$ be so that $A*B$ is well-defined and let $C$ be another set on $G$. Then,

~~$(A*B)\cap C = \int_A ((sB)\cap C)\,ds$~~

~~so in particular~~

$$\#((A*B)\cap C) = \int_A \underbrace{\#((sB)\cap C)}_{\text{Independent of } A!}\,ds.$$

**Pf** $\chi_{(A*B)\cap C}(g) = \chi_{A*B}(g)\cdot\chi_C(g) = \int_A \chi_{sB}(g)\chi_C(g)\,ds$

$$= \int_A \chi_{(sB)\cap C}(g)\,ds$$

□

(AS,101)

**Thm** Let $H$ be a subgroup of the unimodular group $G$. Let $\mathcal{F}$ be a fund. dom. for $H\backslash G$ and let $A$ be a set on $G$ of volume 1. Then, $\mathcal{F} * A$ is a well-defined fund. dom. for $H\backslash G$.

**Rmk** If $0<\mathrm{vol}(A)<\infty$, use $A' = A^{\text{“}\frac{1}{\mathrm{vol}(A)}\text{”}}$. $\rightsquigarrow \mathcal{F}*A' = (\mathcal{F}*A)^{\text{“}\frac{1}{\mathrm{vol}(A)}\text{”}}$.

~~Pf~~ $\cancel{\mathcal{F} \text{ fund. dom.} \iff \sum_{h\in H} h\mathcal{F} = G}$

~~Well-def.: $\chi_{\mathcal{F}}(g)\leq 1\ \forall g\in G$ $\Rightarrow \chi_{\mathcal{F}}$ bounded $+\ \mathrm{vol}(A)<\infty$ $\Rightarrow$ well-def. [and $\mathrm{vol}(\mathcal{F}*A) = \mathrm{vol}(\mathcal{F})$]~~

$\Downarrow \qquad \Downarrow$

~~$\mathcal{F}t$ fund. dom. $\forall t\in G$ $\iff \sum_{h\in H} h\mathcal{F}t = G$~~

$\Downarrow \qquad \Downarrow$

~~“average” $\mathcal{F}*A = \int_A (\mathcal{F}t)\,dt$ fund. dom. $\iff \sum_{h\in H}\int_A (h\mathcal{F}t)\,dt$~~

$$\cancel{= \int_A \underbrace{\Big(\sum_{h\in H} h\mathcal{F}t\Big)}_{G} dt \overset{\mathrm{vol}(A)=1}{=} G} \qquad \square$$

**Pf** Well-definedness:

$\mathcal{F}$ fund. dom. $\Rightarrow \chi_{\mathcal{F}}(g)\leq 1\ \forall g$ and $\mathrm{vol}(A)<\infty$.

Fund. dom.:

Idea: $\mathcal{F}t$ is a fund. dom. for any $t\in G$.

$\Rightarrow$ The “average” $\mathcal{F}*A = \int_A \mathcal{F}t\,dt$ is a fund. dom.

Formally: Let $g\in G$. $\Rightarrow \sum_{h\in H}\chi_{\mathcal{F}*A}(hg) = \sum_{h\in H}\int_G \chi_{\mathcal{F}}(hgt^{-1})\chi_A(t)\,dt$

$$= \int_G \sum_{h\in H}\chi_{\mathcal{F}}(hgt^{-1})\chi_A(t)\,dt = \int_G \chi_A(t)\,dt = \mathrm{vol}(A) = 1. \qquad \square$$

Before continuing with the computation of vol( fund. dom. of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$)

<u>Lemma</u> Fix some $n \geq 1$. Let $Q \subset SL_n(\mathbb{R})$ be a compact and let $C>0$ be a constant. For any $q \in Q$ and $a \in A$ with ($a_i > 0$ and) $a_{i+1} \geq C a_i$ for $i=1,\dots,n-1$, consider the full lattice

$$a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

$[a_1 \ll_C \dots \ll_C a_n]$

$\Lambda = (qa)^{-1}\mathbb{Z}^n = a^{-1}q^{-1}\mathbb{Z}^n$. Its (Euclidean) succ. min. $\lambda_1 \leq \dots \leq \lambda_n$ satisfy $\lambda_i \asymp_{Q,C} a_{n+1-i}^{-1}$ for $i=1,\dots,n$.

indep. of $q, a$

$$(\lambda_1 \asymp a_n^{-1}, \dots, \lambda_n \asymp a_1^{-1}).$$

<u>Pf</u> By Minkowski's second thm,

$$\lambda_1 \cdots \lambda_n \asymp_n \operatorname{covol}(\Lambda) = |\det(a^{-1}q^{-1})| = a_1^{-1} \cdots a_n^{-1}.$$

$\Rightarrow$ It suffices to show $\lambda_i \ll a_{n+1-i}^{-1}$.

Since $Q \subset SL_n(\mathbb{R})$ is compact, the $i$-th row vector of $q^{-1}$ has length $O_Q(1)$. $\Rightarrow$ The $i$-th row vector of $a^{-1}q^{-1}$ has length $O_Q(a_i^{-1})$. $\Rightarrow$ After reordering, the result then follows from

$$a_n^{-1} \ll_C \dots \ll_C a_1^{-1}.$$

The $n$ row vectors are of course linearly independent.

$\square$

To complete the computation of the volume of a fund. dom. of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$, it remains to prove the following thm:

**Thm** Let $\mathcal{F}$ be a measurable fund. dom. for $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$. For $T>0$, let $\mathcal{F}_T = (0, T^{\frac{1}{n}}]\cdot \mathcal{F} \subset GL_n^+(\mathbb{R})$. Then,

$$\#(\mathcal{F}_T \cap M_n(\mathbb{Z})) \sim \overset{\text{Lebesgue!}}{\mathrm{vol}^+}(\mathcal{F}_T) \quad \text{for } T\to\infty.$$

**Pf**

**Thm** let $\mathcal{F}$ be a measurable fund. dom. for $SL_n(\mathbb{Z})\backslash GL_n^+(\mathbb{R})$ (det > 0). For $T>0$, consider the fund. dom. $\mathcal{F}_T = \mathcal{F} \cap GL_n^T(\mathbb{R})$ ($0<\det\le T$) of $SL_n(\mathbb{Z})\backslash GL_n^T(\mathbb{R})$. Then,

$$\#(\mathcal{F}_T \cap M_n(\mathbb{Z})) \sim \overset{\text{Lebesgue}}{\mathrm{vol}^+}(\mathcal{F}_T) \quad \text{for } T\to\infty.$$

**Pf** Both sides are independent of the choice of the fund. dom. $\mathcal{F}_T$. $\Rightarrow$ We may w.l.o.g. assume that the support of $\mathcal{F}_T$ is contained in $\tilde{\mathcal{F}}_{\text{Siegel}} \subset GL_n(\mathbb{R})$.

Let $S$

**Thm** Let $\mathcal{F}$ be a measurable fund. dom. for $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$.

For $T>0$, consider the fund. dom. $\mathcal{F}_T = (0,T]\cdot\mathcal{F}$ for $SL_n(\mathbb{Z})\backslash GL_n^{T^n}(\mathbb{R})$. ($0<\det\le T^n$)

Then,

$$\#(\mathcal{F}_T\cap M_n(\mathbb{Z})) \sim \underset{\text{Lebesgue!}}{vol^+}(\mathcal{F}_T) \quad \text{for } T\to\infty.$$

**Pf** Both sides are indep. of the choice of fund. dom. $\mathcal{F}_T$. (The action of $SL_n(\mathbb{R})$ preserves the Lebesgue measure.)

Assume w.l.o.g. that $\mathrm{supp}(\mathcal{F})\subset \widetilde{\mathcal{F}}^{\,SL_n(\mathbb{R})}_{\text{Siegel}} = N' A_1' K_1$.

[Now, use convolution to make $\mathcal{F}$ nicer!]

Fix any subset $S\subset SL_n(\mathbb{R})$ of volume 1 whose boundary is Lipschitz.

$\Rightarrow$ $\mathcal{F}*S$ is also a fund. dom. for $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$,

$(\mathcal{F}*S)_T = (0,T]\cdot(\mathcal{F}*S)$ is also a fund. dom. for $SL_n(\mathbb{Z})\backslash GL_n^{T^n}(\mathbb{R})$,

with $vol^+((\mathcal{F}*S)_T) = vol^+(\mathcal{F}_T)$.

$\Rightarrow$ It suffices to prove

$$\#((\mathcal{F}*S)_T\cap M_n(\mathbb{Z})) \sim vol^+(\mathcal{F}_T) \quad \text{for } T\to\infty.$$

But $$LHS = \int_{\mathcal{F}} \underbrace{\#((0,T]\cdot gS\cap M_n(\mathbb{Z}))}_{=:f(g)}\, dg = \int_{\mathcal{F}} f(g)\,dg.$$

Now, we ~~want to~~ apply Widmer's Thm. to the integrand.

Write $g = nak$ with $n\in N'$, $a=\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}\in A_1'$,

$k\in K_1 = SO_n(\mathbb{R})$. The set $gS$ could be narrow and long if $a_1$ is small and $a_n$ is large!

$\rightsquigarrow$ It'll be better to rescale the lattice $M_n(\mathbb{Z})$ than the set $S$.

$$f(g) = \#((0,T]\cdot \underset{nak}{g}S\cap M_n(\mathbb{Z})) = \#((0,T]\cdot kS\cap (na)^{-1}M_n(\mathbb{Z}))$$