Math 223a: Algebraic Number Theory

Fall 2020

Problem set #7

due Friday, October 23 at 10:30am

Problem 1. Let K be a finite Galois extension of $\mathbb{Q}$. Show that the Galois group $\mathrm{Gal}(K|\mathbb{Q})$ is generated by the inertia subgroups $I(\mathfrak{p}|p)$ for primes $\mathfrak{p} \mid p$.

Problem 2. Let $p_1, \ldots, p_k \equiv 3 \mod 4$ be distinct prime numbers with k odd.

a) Show that the Hilbert class field of $K = \mathbb{Q}(\sqrt{-p_1 \cdots p_k})$ contains $L = \mathbb{Q}(\sqrt{-p_1}, \ldots, \sqrt{-p_k})$. (Note that by class field theory, this implies that the class number of K is divisible by $[L : K] = 2^{k-1}$.)

b) Write $(p_i) = \mathfrak{p}_i^2$ in $\mathcal{O}_K$. Show that $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ generate a subgroup of Cl_K of order 2^{k-1}.

Problem 3. Let $L|K$ be an unramified degree n extension of local fields and let $\mathbb{F}_{q^n}|\mathbb{F}_q$ be the corresponding extension of residue fields. Show the following facts without using class field theory:[1]

a) The norm map $\mathrm{Nm}_{\mathbb{F}_{q^n}|\mathbb{F}_q} : \mathbb{F}_{q^n}^\times \to \mathbb{F}_q^\times$ is surjective.

b) The trace map $\mathrm{Tr}_{\mathbb{F}_{q^n}|\mathbb{F}_q} : \mathbb{F}_{q^n} \to \mathbb{F}_q$ is surjective.

c) The norm map $\mathrm{Nm}_{L|K} : \mathcal{O}_L^\times \to \mathcal{O}_K^\times$ is surjective. (Hint: Imitate a proof of Hensel's lemma and use a) and b).)

d) The image of the norm map $\mathrm{Nm}_{L|K} : L^\times \to K^\times$ is the subset $\{x \in K^\times \mid v_K(x) \equiv 0 \mod n\}$ of $K^\times$ (which corresponds to the subset $\mathcal{O}_K^\times \times n\mathbb{Z}$ of $\mathcal{O}_K^\times \times \mathbb{Z}$).

[1] But also think about why c) and d) follow from class field theory!