Math 223a: Algebraic Number Theory

Fall 2020

Problem set #6

due Friday, October 16 at 10:30am

Problem 1. Show the quadratic reciprocity law:

a) For any odd prime p, we have $(\frac{2}{p}) = +1$ if and only if $p \equiv \pm 1 \mod 8$.

b) For any odd primes $p \neq q$, we have $(\frac{p}{q}) = (\frac{q}{p})$ if and only if $p \equiv 1 \mod 4$ or $q \equiv 1 \mod 4$.

(Hint: Use that $\mathbb{Q}(\sqrt{q}) \subseteq \mathbb{Q}(\zeta_n)$ for an appropriate number n computed in class.)

Problem 2. For any prime number p, show that $\mathrm{Gal}(\bigcup_n \mathbb{Q}_p(\zeta_{p^n})|\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$.

Problem 3. Let G be a commutative topological group which is compact and such that $\bigcap_{U \subseteq G \text{ open subgroup}} U = 0$. Show that the natural map $G \to \widehat{G}$ into its profinite completion is an isomorphism.

Problem 4. Let R be a (commutative) topological ring. Identifying the set $M_n(R)$ of $n \times n$-matrices with R^{n^2} (by sending a matrix to its entries), we obtain a topology on $M_n(R)$. (This makes $M_n(R)$ a topological ring.) Show that $\mathrm{SL}_n(R) \subseteq M_n(R)$ is a topological group with the subspace topology.

Problem 5. a) Show that the image of $\mathbb{Q}^\times \to (\mathbb{A}_\mathbb{Q}^S)^\times$ is not dense for any finite set of places S. (In other words, the multiplicative group $\mathbb{G}_m$ doesn't satisfy strong approximation over $\mathbb{Q}$ away from S.)

b) Show that the image of $\mathrm{SL}_n(\mathbb{Q}) \to \mathrm{SL}_n(\mathbb{A}_\mathbb{Q}^S)$ is dense for every nonempty set of places S. (In other words, SL_n satisfies strong approximation over $\mathbb{Q}$ away from S.)

(Note, however, that $R^\times \subset \mathrm{SL}_2(R)$ for any (commutative) ring R.)