Math 223a: Algebraic Number Theory

Fall 2020

Problem set #3

due Friday, September 25 at 10:30am

Problem 1. a) Show that every subgroup H of $\mathbb{Z}_p^\times$ of finite index is open.

b) Show that every subgroup H of $\mathbb{Q}_p^\times$ of finite index is open.

Problem 2. Let K be a local field with residue field $\mathbb{F}_q$. Show that K has exactly one unramified extension of any degree $n \geqslant 1$, namely the field L obtained by adjoining all $(q^n - 1)$-th roots of unity.

Problem 3. Let K be complete with respect to a discrete valuation v. Let $f_1, \ldots, f_n \in \mathcal{O}_v[X_1, \ldots, X_n]$ be n polynomials in n variables. Assume that $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \in \kappa_v^n$ is a root of each $f_i \mod \mathfrak{p}_v$, but not a root of the Jacobian determinant $\det \left(\frac{\partial f_i}{\partial X_j}\right)_{i,j} \mod \mathfrak{p}_v$. Then, there is exactly one common root $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_v^n$ of all $f_1, \ldots, f_n$ such that $\alpha \equiv \overline{\alpha} \mod \mathfrak{p}_v$.

Problem 4. Fix a prime number $p > 2$ and let

$$f(X, Y) = \sum_{\substack{i,j \geqslant 0: \\ i+j \leqslant p-2}} X^i Y^j.$$

For which pairs $(a, b) \in \mathbb{R} \times \mathbb{R}$ does there exist a pair $(x, y) \in \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$ satisfying

$$f(px, py) = f(px^{-1}, py^{-1}) = 0$$

with $v_p(x) = a$ and $v_p(y) = b$?

Problem 5. Let the field K be complete with respect to the normalized discrete valuation v. Let $f(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n]$ be a polynomial. Let $w_1, \ldots, w_n \in \mathbb{Q}$. To any monomial $M = c \cdot X_1^{a_1} \cdots X_n^{a_n}$ in f, associate the number $u(M) = v(c) + a_1 w_1 + \cdots + a_n w_n$. Assume that $u(M) \geqslant 0$ for all monomials M in f and that $u(M) = 0$ for at least two monomials M in f. Show that there exist $x_1, \ldots, x_n \in \overline{K}^\times$ with valuations $v(x_i) = w_i$ (for all i) that satisfy $f(x_1, \ldots, x_n) = 0$.