Math 223a: Algebraic Number Theory

Fall 2020

Problem set #12

due Monday, November 30 at 10:30am

Problem 1. Let $L|K$ be a finite Galois extension of nonarchimedean local fields with Galois group G and ramification index e. Show that $H^1(G, \mathcal{O}_L^\times)$ is isomorphic to $\mathbb{Z}/e\mathbb{Z}$. (Hint: Use Hilbert 90.)

Problem 2. Give an example of a finite Galois extension $K|\mathbb{Q}$ of number fields such that $\mathcal{O}_K$ does not have a normal $\mathbb{Z}$-basis: There is no $x \in \mathcal{O}_L$ such that $(g(x))_{g \in \mathrm{Gal}(L|K)}$ forms a $\mathbb{Z}$-module basis of $\mathcal{O}_L$.

Problem 3. Prove the normal basis theorem for finite fields: Any extension $\mathbb{F}_{q^n}|\mathbb{F}_q$ of finite fields has a basis of the form $\{g(x) \mid g \in \mathrm{Gal}(\mathbb{F}_{q^n}|\mathbb{F}_q)\}$ with $x \in \mathbb{F}_{q^n}$. (Hint: Make $\mathbb{F}_{q^n}$ a $\mathbb{Z}[T]$-module by letting $T.x = x^q$ for $x \in \mathbb{F}_q$. Apply the fundamental theorem of finitely generated modules over principal ideal domains.)

Let G be a finite group and let K be a field with trivial G-action. The abelian groups $H^i(G, K)$ have a canonical K-vector space structure: simply apply the cohomology functor $H^i(G, -)$ to the scalar multiplication maps on K to obtain the scalar multiplication maps on $H^i(G, K)$.

Problem 4. Show that each of the vector spaces $H^i(G, K)$ $(i \geqslant 0)$ is finite-dimensional.

Problem 5 (bonus)**.** Let G_1 and G_2 be finite groups and let K be a field with trivial G-action. Construct an isomorphism of K-vector spaces

$$H^n(G_1 \times G_2, K) \cong \sum_{i+j=n} H^i(G_1, K) \otimes_K H^j(G_2, K).$$

(This is called the *Künneth formula.*)

Problem 6 (bonus)**.** Let $G = (\mathbb{Z}/2\mathbb{Z})^k$ for $k \geqslant 1$. Compute the dimension of $H^i(G, \mathbb{F}_2)$ as an $\mathbb{F}_2$-vector space for all $i \geqslant 0$.