Math 223a: Algebraic Number Theory

Fall 2020

Problem set #10

due Friday, November 13 at 10:30am

Problem 1. Let R be any ring and let $f(X) = a_1X + a_2X^2 + \cdots \in R[[X]]$ be a power series. Show that the following are equivalent:

i) We have $a_1 \in R^\times$.

ii) There exists a power series $g(X) = b_1X + b_2X^2 + \cdots \in R[[X]]$ such that $f(g(X)) = g(f(X)) = X$.

Problem 2. Let F be a formal group over a ring R (as defined in class).

a) Show that $F(X, 0) = F(0, X) = X$.

b) Show that there is a power series $i(X) \in R[[X]]$ with

$$i(X) = -X + (\text{terms of degree} \geqslant 2)$$

and $F(X, i(X)) = 0$.

Problem 3. Let R be an integral domain with field of fractions K. Show that the set $\mathrm{End}_R(\mathbb{G}_a)$ of formal endomorphisms $f(X) \in R[[X]]$ of the additive formal group $\mathbb{G}_a$ consists exactly of the following elements:

a) The polynomials aX for $a \in R$ if $\mathrm{char}(K) = 0$.

b) The power series $\sum_{k=0}^{\infty} a_k X^{p^k}$ for $a_0, a_1, \dots \in R$ if $\mathrm{char}(K) = p \neq 0$.

Problem 4 (bonus)**.** Let R be a commutative $\mathbb{Q}$-algebra.

a) Let $F(X, Y) \in R[[X, Y]]$ be any formal group over R. Show that there is a unique isomorphism $\log_F : F \to \mathbb{G}_a$ of formal groups over R with $\log_F(X) = X + (\text{deg.} \geqslant 2)$. (Hint: You're trying to make $\log_F(F(X, Y)) = \log_F(X) + \log_F(Y)$. Try differentiating both sides with respect to Y. Also differentiate the associativity law with respect to Y.)

b) Show that $\log_{\mathbb{G}_m}(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot X^n$ defines an isomorphism $\log_{\mathbb{G}_m} : \mathbb{G}_m \to \mathbb{G}_a$.