Math 223a: Algebraic Number Theory

Fall 2020

Problem set #1

due Friday, September 11 at 10:30am

Problem 1. Let $n \geqslant 1$ and $a \in \mathbb{Z}$. Show that the equation $x^n = a$ has a solution x in $\mathbb{Z}$ if and only if it has a solution in $\mathbb{R}$ and a solution in $\mathbb{Z}_p$ for all primes p. (Hence, the equation $x^n = a$ satisfies the local-global principle.)

Problem 2. Consider the number field $K = \mathbb{Q}(\sqrt{-5})$.

a) Show that the ideal (2) of K ramifies: $(2) = \mathfrak{p}^2$ for some prime ideal $\mathfrak{p}$.

b) Show that $\mathfrak{p}$ is not a principal ideal.

c) Find a generator of the maximal ideal of the localization of $\mathcal{O}_K$ at $\mathfrak{p}$.

Problem 3. Consider a nonzero element a of a finite field $\mathbb{F}_q$ and any integer $n \geqslant 1$. Show that $a = x^n$ for some $x \in \mathbb{F}_q$ if and only if $a^{(q-1)/\gcd(q-1,n)} = 1$. (Hint: Use that the group $\mathbb{F}_q^\times$ is cyclic.)

Problem 4. Let L be a Galois extension of $\mathbb{Q}$ with Galois group S_3. Let K_3 be the degree 3 extension of $\mathbb{Q}$ fixed by $\langle(2\ 3)\rangle \subset S_3$. Let K_2 be the degree 2 extension fixed by $\langle(1\ 2\ 3)\rangle \subset S_3$. Show that no prime p of $\mathbb{Q}$ is inert in both K_3 and K_2.

Problem 5. Consider any polynomial $f(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$ and any prime number p. Assume that for all $k \geqslant 0$, there exist $x_1, \ldots, x_n \in \mathbb{Z}$ such that $f(x_1, \ldots, x_n) \equiv 0 \mod p^k$. Show that there exist $y_1, \ldots, y_n \in \mathbb{Z}_p$ such that $f(y_1, \ldots, y_n) = 0$.