1.1. Valuations

Def Let K be a field. A valuation on K is a map

$v: K \longrightarrow \mathbb{R} \cup \{\infty\}$ such that:

a) $v(x) = \infty \Leftrightarrow x = 0$

b) $v(xy) = v(x) + v(y)$ (i.e. $v: K^\times \to \mathbb{R}$ is a group hom.)

c) $v(x+y) \geq \min(v(x), v(y))$.

It is discrete if

d) $v(K^\times) = s \cdot \mathbb{Z} \subset \mathbb{R}$ for some $s \geq 0$.

(i.e. $v(K^\times) \subset \mathbb{R}$ is a discrete subgroup)

It is a normalized discrete valuation if

e) $v(K^\times) = \mathbb{Z}$. Then, any $\pi \in K^\times$ with $v(\pi) = 1$ is called a uniformizer. We denote one of them by π_v.

Ex Trivial valuation: $v(x) = 0 \quad \forall x \in K^\times$

Rmk If v is a (disc.) val., then so is $\lambda \cdot v$ for any $\lambda > 0$.

Main example If $\mathcal{O}_K$ is a Dedekind domain and $\mathfrak{p}$ is a prime (= nonzero prime ideal), then

$$v_\mathfrak{p}(x) = \sup \{ n \in \mathbb{Z} \mid x \in \mathfrak{p}^n \}$$

= number of times x is divisible by $\mathfrak{p}$

= exponent of $\mathfrak{p}$ in the factorization of (x)

defines a normalized discrete valuation on K, called the $\mathfrak{p}$-adic valuation.

Rmk Any valuation satisfies:

i) $v(1) = v(-1) = 0$

ii) If $v(x) \neq v(y)$, then equality holds in c):

$$v(x+y) = \min(v(x), v(y)).$$

Pf i) grp. hom. $\Rightarrow v(1) = 0$

$$(-1)^2 = 1 \Rightarrow 2v(-1) = v(1) = 0$$

ii) Say $v(x) < v(y)$ and assume

$$v(x+y) > \min(v(x), v(y)) = v(x)$$

$$\Rightarrow v(x) = v((x+y)+(-y)) \underset{c)}{\geq} \min(v(x+y), \underbrace{v(-y)}_{v(-1)+v(y)=v(y)}) > v(x)$$

$\lightning$

$\square$

Def/thm Let v be a valuation. Then

$\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\}$ is a local ring
(the valuation ring)

with field of fractions K, unit group
$\mathcal{O}_v^\times = \{x \in K \mid v(x) = 0\}$, (unique) maximal ideal
$\mathfrak{p}_v := \{x \in K \mid v(x) > 0\}$, and residue field
$\kappa_v := \mathcal{O}_v / \mathfrak{p}_v$.

Thm If v is a normalized disc. val., then
$\mathcal{O}_v$ is a PID: Any ideal is of the form

$$\{x \in K \mid v(x) \geq n\} = \mathfrak{p}_v^n = (x_0) \quad \text{for some } n \geq 0$$

for any $x_0 \in K$ with $v(x_0) = n$.
In particular, $\mathfrak{p}_v = (\pi_v)$.

Pf Consider any ideal I. Let $n = \min_{x \in I} v(x)$ and

choose any $x_0' \in I$ with $v(x_0') = n$. Then,

$$I \supseteq (x_0') = \{x \in K \mid v(x) \geq n\} \supseteq I,$$
so $I = (x_0')$. For any $x_0 \in K$ with $v(x_0) = n$,
we have $v\left(\frac{x_0}{x_0'}\right) = 0$, so $\frac{x_0}{x_0'} \in \mathcal{O}_v^\times$, so
$(x_0) = (x_0') = I$.

In particular $\mathfrak{p}_v = (\pi_v)$.

$\Rightarrow \mathfrak{p}_v^n = (\pi_v^n)$ and $v(\pi_v^n) = n$. $\square$

<u>Lemma</u> If v is the $\mathfrak{p}$-adic valuation for a prime $\mathfrak{p}$ in a Ded. dom. $\mathcal{O}_K$, then $\mathcal{O}_v$ is the <u>localization</u> of $\mathcal{O}_K$ at $\mathfrak{p}$ and we have $\mathfrak{p}_v = \mathfrak{p}\mathcal{O}_v$ and $\mathcal{O}_v/\mathfrak{p}_v^n \cong \mathcal{O}_K/\mathfrak{p}^n$

$x \bmod \mathfrak{p}_v^n \longleftarrow\!\!\shortmid\ x \bmod \mathfrak{p}^n$

for any $n \geq 0$. Also, v is the $\mathfrak{p}_v$-adic valuation.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \hat{} \ \mathcal{O}_v$

<u>Ex</u> $\mathcal{O}_K = \mathbb{Z}$, $K = \mathbb{Q}$, $v = v_p$ (p-adic val.)

$\Rightarrow \mathcal{O}_v = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ b \not\equiv 0 \bmod p \right\} \subset \mathbb{Q}$

$\mathcal{O}_v^\times = \mathbb{Z}_{(p)}^\times = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ a, b \not\equiv 0 \bmod p \right\} \subset \mathbb{Q}$

$\mathfrak{p}_v = p\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ a \equiv 0 \bmod p,\ b \not\equiv 0 \bmod p \right\} \subset \mathbb{Q}$

$\pi_v =$ for example p, $-p$.

$$\mathbb{Z}_{(p)}/p^n\mathbb{Z}_{(p)} \cong \mathbb{Z}/p^n\mathbb{Z}$$

$$x \longleftarrow\!\!\shortmid\ x$$

$$\frac{a}{b} \longmapsto a \cdot b^{-1} \bmod p^n$$

(note that b is invertible mod p^n because $b \not\equiv 0 \bmod p$)

Lemma Let v be a norm. disc. val.

Look at the filtration

$$\mathcal{O}_v \supseteq \mathfrak{p}_v \supseteq \mathfrak{p}_v^2 \supseteq \dots$$

We have $\mathfrak{p}_v^a / \mathfrak{p}_v^b \cong \mathcal{O}_v / \mathfrak{p}_v^{b-a}$ as groups

$$\pi_v^a \cdot x \longleftarrow\!\shortmid \; x$$

for any $a \le b$.

Rmk The isom. depends on π_v.

Lemma Let v be a norm. disc. val.

Look at the filtration

$$\mathcal{O}_v^\times \supseteq U^{(1)} \supseteq U^{(2)} \supseteq \dots$$

with $U^{(n)} = 1 + \mathfrak{p}_v^n$.

We have

a) $\mathcal{O}_v^\times / U^{(n)} \cong (\mathcal{O}_v / \mathfrak{p}_v^n)^\times$ as a group

$$x \cdot U^{(n)} \longmapsto x \bmod \mathfrak{p}_v^n$$

b) $U^{(n)} / U^{(n+1)} \cong \mathcal{O}_v / \mathfrak{p}_v = k_v$ as a group

$$1 + \pi_v^n x \longleftarrow\!\shortmid \; x$$

Rmk The isom. in b) depend on π_v.

Pf b) $f : \mathcal{O}_v \longrightarrow U^{(n)} / U^{(n+1)}$ is a group homomorphism:

$$x \longmapsto 1 + \pi_v^n x$$

$$\frac{f(x)\, f(y)}{f(x+y)} = \frac{(1+\pi^n x)(1+\pi^n y)}{1 + \pi^n (x+y)}$$

$$= \frac{1+\pi^n(x+y)+\pi^{2n}xy}{1+\pi^n(x+y)} = 1+\frac{\pi^{2n}xy}{1+\pi^n(x+y)}$$

$$\equiv 1 \mod \mathfrak{P}_v^{n+1}.$$

$$\Rightarrow \frac{f(x)f(y)}{f(x+y)} \in U^{(n+1)} \Rightarrow f \text{ is a grp. hom.}$$

f is clearly surj. because $\mathfrak{P}_v^n = (\pi_v^n)$.

$\ker(f) = \mathfrak{P}_v$.

$\square$

Let's see what valuations there are in a few examples of fields K!

<u>Thm 1.1</u> Any normalized disc. val. v on $\mathbb{Q}$ is of the form $v = v_p$ for some prime number p.

<u>Pf</u> For $x = \pm \prod_p p^{e_p} \in \mathbb{Q}^\times$, we have

$$v(x) = \sum_p e_p \cdot v(p).$$

$\Rightarrow$ Valuation is determined by $v(p)$ for the prime numbers p.

$\mathcal{O}_v = \{x \in \mathbb{Q} \mid v(x) \geq 0\}$ is a subring of $\mathbb{Q}$, so $\mathbb{Z} \subset \mathcal{O}_v$. $\Rightarrow v(p) \geq 0$.

$\mathfrak{p}_v \cap \mathbb{Z} = \{x \in \mathbb{Z} \mid v(x) > 0\}$ is a prime ideal of $\mathbb{Z}$.

$\Rightarrow v(p) > 0$ for (at most) one prime number p and $v(q) = 0$ for all $q \neq p$.

v normalized $\Rightarrow v(p) = 1$. $\Rightarrow v = v_p$ (p-adic val.)

$\square$

<u>Thm</u> A finite field $\mathbb{F}_q$ has no nontriv. val.

<u>Pf</u> For any $x \in \mathbb{F}_q^\times$, $x^{q-1} = 1$.

$\Rightarrow (q-1)v(x) = v(1) = 0$. $\Rightarrow v(x) = 0$. $\square$

<u>Thm</u> An algebraically closed field K has no nontriv. disc. val.

<u>Pf</u> Assume v is a norm. disc. val.

$v(\pi_v) = 1$. $\Rightarrow v(\sqrt{\pi_v}) = \frac{1}{2} \notin \mathbb{Z}$.

$\Rightarrow$ not normalized. $\square$

<u>Thm</u> Let k be a field that has no nontriv. disc. val. Then, the norm. disc. val. of $K = k(T)$ are:

- $v = v_{f(T)}$, the $f(T)$-adic val. for some irred. monic pol. $f(T) \in k[T]$.

the Dedekind dom.

- $v = v_{deg}$ given by

$$v_{deg}\left(\frac{a(T)}{b(T)}\right) = \deg(b(T)) - \deg(a(T))$$

for $a(T), b(T) \in k(T)$.

Rmk $v_{\deg}$ is the $(\frac{1}{T})$-adic val. for the ideal $(\frac{1}{T})$ of the Ded. dom. $k[\frac{1}{T}]$.

Pf of rmk

Write $a(T) = T^{\deg(a)} \cdot \tilde{a}(T)$

$b(T) = T^{\deg(b)} \cdot \tilde{b}(T)$

with $\tilde{a}(T), \tilde{b}(T) \in k[\frac{1}{T}]$ with nonzero const. coeff.

$v_{\frac{1}{T}}(a(T)) = -\deg(a) + 0 = -\deg(a)$

$v_{\frac{1}{T}}(b(T)) = -\deg(b)$.

$\Rightarrow v_{\frac{1}{T}}\left(\frac{a(T)}{b(T)}\right) = \deg(b) - \deg(a)$.

$\square$