Math 223a: Algebraic Number Theory

Fall 2019

Homework #9

due Tuesday, November 12 at noon

Problem 1. Let G be a finite cyclic group generated by σ. Show that the following sequence is exact, where $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is the map sending $\sum_{g\in G} a_g g$ to $\sum_{g\in G} a_g$ and we denote the multiplication by $\sigma - 1$ map on $\mathbb{Z}[G]$ by $\sigma - 1$ and the multiplication by $N = \sum_{g\in G} g$ map on $\mathbb{Z}[G]$ by N:

$$0 \longleftarrow \mathbb{Z} \xleftarrow[\varepsilon]{} \mathbb{Z}[G] \xleftarrow[\sigma-1]{} \mathbb{Z}[G] \xleftarrow[N]{} \mathbb{Z}[G] \xleftarrow[\sigma-1]{} \mathbb{Z}[G] \xleftarrow[N]{} \cdots$$

Problem 2. Let G be a finite cyclic group of order n and let A be an abelian group with trivial G-action. Show that $H^0(G, A) = A$, $H^i(G, A) = \{a \in A \mid na = 0\}$ for odd $i \geqslant 1$, and $H^i(G, A) = A/nA$ for even $i \geqslant 2$.

Problem 3. Let G be a finite group and A an abelian group with trivial G-action. Show that $H^1(G, A)$ is isomorphic to $\mathrm{Hom}_{\mathrm{grp}}(G, A)$, the group of group homomorphisms $G \to A$ (with addition given by $(f_1 + f_2)(g) = f_1(g) + f_2(g)$).

Let G be a finite group and let K be a field with trivial G-action. The abelian groups $H^i(G, K)$ have a canonical K-vector space structure: simply apply the cohomology functor $H^i(G, -)$ to the scalar multiplication maps on K to obtain the scalar multiplication maps on $H^i(G, K)$.

Problem 4. Show that each of the vector spaces $H^i(G, K)$ $(i \geqslant 0)$ is finite-dimensional.

Problem 5 (bonus)**.** Let G_1 and G_2 be finite groups and let K be a field with trivial G-action. Construct an isomorphism of K-vector spaces

$$H^n(G_1 \times G_2, K) \cong \sum_{i+j=n} H^i(G_1, K) \otimes_K H^j(G_2, K).$$

(This is called the *Künneth formula*.)

Problem 6. Let $G = (\mathbb{Z}/2\mathbb{Z})^k$ for $k \geqslant 1$. Compute the dimension of $H^i(G, \mathbb{F}_2)$ as an $\mathbb{F}_2$-vector space for all $i \geqslant 0$. (You may use the result from the bonus problem.)