Math 223a: Algebraic Number Theory

Fall 2019

Homework #7

due Tuesday, October 29 at noon

Problem 1. Let $L|K$ be a Galois extension of local fields and let $\sigma \in \text{Gal}(L|K)$ be any automorphism. Let $a \in \mathcal{O}_L$ and $b \in \mathfrak{p}_L$. Show that the equation $x - b\sigma(x) = a$ has a unique solution $x \in \mathcal{O}_L$.

Recall that the additive formal group is $\mathbb{G}_a(X, Y) = X + Y$ and the multiplicative formal group is $\mathbb{G}_m = (X + 1)(Y + 1) - 1 = X + Y + XY$.

Problem 2.

a) Let R be an integral domain of characteristic zero (meaning $\mathbb{Z} \subseteq R$). Show that the ring $\text{End}_R(\mathbb{G}_a)$ of formal endomorphisms $f(X) \in R[[X]]$ of the additive formal group consists exactly of the polynomials aX for $a \in R$.

b) Let R be an integral domain of characteristic $p \geqslant 2$ (meaning p is zero in R). Show that the ring $\text{End}_R(\mathbb{G}_a)$ of formal endomorphisms $f(X) \in R[[X]]$ of the additive formal group consists exactly of the power series $\sum_{k=0}^{\infty} a_k X^{p^k}$ for $a_0, a_1, \dots \in R$.

Problem 3. Consider two different uniformizers π, π' of a local field K. Show that $K_{\pi,n} \neq K_{\pi',n}$ for sufficiently large $n \geqslant 1$. (In particular, if we call the corresponding Lubin-Tate modules F, F', this means that $F(n)$ and $F'(n)$ are nonisomorphic formal $\mathcal{O}_K$-modules.)

Problem 4. Let $L|K$ be a nontrivial abelian extension of local fields. Show that $L^{\text{ab}} \neq K^{\text{ab}}$. You may assume the Hasse–Arf theorem for this problem.

Problem 5. Let R be a commutative $\mathbb{Q}$-algebra.

a) Let $F(X, Y) \in R[[X, Y]]$ be any formal group over R. Show that there is a unique isomorphism $\log_F : F \to \mathbb{G}_a$ of formal groups over R with $\log_F(X) = X + (\text{deg.} \geqslant 2)$. (Hint: You're trying to make $\log_F(F(X, Y)) = \log_F(X) + \log_F(Y)$. Try differentiating both sides with respect to Y. Also differentiate the associativity law with respect to Y.)

b) Show that $\log_{\mathbb{G}_m}(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot X^n$ defines an isomorphism $\log_{\mathbb{G}_m} : \mathbb{G}_m \to \mathbb{G}_a$.