Math 223a: Algebraic Number Theory

Fall 2019

Homework #4

due Thursday, October 3 at noon

Problem 1. Let $L|K$ be an unramified extension of local fields and let $\mathbb{F}_{q^n}|\mathbb{F}_q$ be the corresponding extension of residue fields. Show that $L = K(\zeta_{q^n-1})$.

Problem 2. Let K be a local field. Equip $\mathbb{Z}$ with the discrete topology. Show that the group isomorphism $\mathcal{O}_K^\times \times \mathbb{Z} \to K^\times$ sending (x, n) to $x\pi_K^n$ is a homeomorphism.

Definition. Let K be a local field. A polynomial $f(X) = a_n X^n + \cdots + a_0 \in K[X]$ is called an *Eisenstein polynomial* if $v_K(a_n) = 0$, $v_K(a_{n-1}) \geqslant 1$, $\ldots$, $v_K(a_1) \geqslant 1$, and $v_K(a_0) = 1$.

Problem 3. Let K be a local field with residue field $\kappa_K \cong \mathbb{F}_q$. Consider an Eisenstein polynomial $f(X) \in K[X]$ of degree $q-1$. Let $\alpha \in \overline{K}$ be a root of $f(X)$ and $L = K(\alpha)$.

a) Show that L is a Galois extension of K.

b) What is the Galois group of L over K?

Problem 4. Let $L|K$ be an unramified degree n extension of local fields and let $\mathbb{F}_{q^n}|\mathbb{F}_q$ be the corresponding extension of residue fields.

a) Show that the norm map $\mathrm{Nm}_{\mathbb{F}_{q^n}|\mathbb{F}_q} : \mathbb{F}_{q^n}^\times \to \mathbb{F}_q^\times$ is surjective.

b) Show that the trace map $\mathrm{Tr}_{\mathbb{F}_{q^n}|\mathbb{F}_q} : \mathbb{F}_{q^n} \to \mathbb{F}_q$ is surjective.

c) Show that the norm map $\mathrm{Nm}_{L|K} : \mathcal{O}_L^\times \to \mathcal{O}_K^\times$ is surjective.

d) Show that the image of the norm map $\mathrm{Nm}_{L|K} : L^\times \to K^\times$ is the subset $\{x \in K^\times \mid v_K(x) \equiv 0 \mod n\}$ of $K^\times$ (which corresponds to the subset $\mathcal{O}_K^\times \times n\mathbb{Z}$ of $\mathcal{O}_K^\times \times \mathbb{Z}$).

Problem 5. Let K be a local field. Consider the projective limit

$$\varprojlim_{U \subseteq K^\times \text{ open subgroup of finite index}} K^\times/U,$$

the set of tuples $(x_U)_U \in \prod_U K^\times/U$ such that $x_U U = x_V U$ for all $U \supseteq V$. Show that

$$\varprojlim_U K^\times/U \cong \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}.$$