Math 223a: Algebraic Number Theory

Fall 2019

Homework #3

due Thursday, September 26 at noon

Problem 1. Let $L|K$ be a finite extension of number fields. Show that the following two statements are equivalent:

a) For any two primes $\mathfrak{P}, \mathfrak{P}'$ in $\mathcal{O}_L$ with $\mathfrak{P} \cap \mathcal{O}_K = \mathfrak{P}' \cap \mathcal{O}_K$, we have $\kappa(\mathfrak{P}) = \kappa(\mathfrak{P}')$. ("Any two prime divisors of a prime $\mathfrak{p}$ in $\mathcal{O}_K$ have the same residue field.")

b) The field extension $L|K$ is a Galois extension.

Problem 2. Let K be an algebraic field extension of $\mathbb{Q}$ of degree $n \geqslant 2$. Show that there are infinitely many prime numbers p that have no prime divisor $\mathfrak{p}$ in $\mathcal{O}_K$ with residue field $\kappa(\mathfrak{p}) = \mathbb{F}_p$.

Problem 3. Let $K \subseteq \mathbb{Q}(\zeta_\infty)$ be a finite field extension of $\mathbb{Q}$. Show that a prime number p divides the conductor of K (smallest $n \geqslant 1$ such that $K \subseteq \mathbb{Q}(\zeta_n)$) if and only if it divides the discriminant of K.

Problem 4. a) Show that every subgroup H of $\mathbb{Z}_p^\times$ of finite index is open.

b) Show that every subgroup H of $\mathbb{Q}_p^\times$ of finite index is open.

Problem 5. Let K be complete with respect to a discrete valuation v. Let $f_1, \ldots, f_n \in \mathcal{O}_v[X_1, \ldots, X_n]$ be n polynomials in n variables. Assume that $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \in \kappa_v^n$ is a root of each $f_i \mod \mathfrak{p}_v$, but not a root of the Jacobian determinant $\det\left(\frac{\partial f_i}{\partial X_j}\right)_{i,j} \mod \mathfrak{p}_v$. Then, there is exactly one common root $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_v^n$ of all $f_1, \ldots, f_n$ such that $\alpha \equiv \overline{\alpha} \mod \mathfrak{p}_v$.