Math 223a: Algebraic Number Theory

Fall 2019

Homework #2

due Thursday, September 19 at noon

> Do we have negative prime numbers? $\ldots, -7, -5, -3, -2, \ldots$
>
> user103028, *Mathematics Stack Exchange*: `https://math.stackexchange.com/questions/1002459/do-we-have-negative-prime-numbers`

Problem 1. Let K be a field extension of $\mathbb{Q}$ and let $n \geqslant 1$. Show that the following are equivalent:

a) For any two prime numbers p, p' such that $p \equiv \pm p' \mod n$, the ideals (p) and (p') split in the same way in $\mathcal{O}_K$.

b) The field K is contained in $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

Problem 2. Which nonzero integers a satisfy the following property?

There exists an integer $n \geqslant 1$ such that any two prime numbers p, p' with $p \equiv \pm p' \mod n$ split in the same way in $\mathbb{Q}(\sqrt{a})$.

Problem 3. Let K be a finite Galois extension of $\mathbb{Q}$ with Galois group G and let $n \geqslant 1$. Show that the following statements are equivalent:

a) For each $r \in (\mathbb{Z}/n\mathbb{Z})^\times$ and each conjugacy class C in G, there exists some prime number $p \equiv r \mod n$ such that $\mathrm{Frob}_{K|\mathbb{Q}}(p) = C$. ("Each unramified splitting behavior occurs in each invertible residue class modulo n.")

b) We have $K \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$.

Problem 4. Show that there is a number field K satisfying the following two properties:

a) There is an integer $n \geqslant 1$ such that for all primes $p \equiv p' \mod n$, there is a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ dividing p with residue field $\kappa(\mathfrak{p}) = \kappa(p) = \mathbb{F}_p$ if and only if there is a prime ideal $\mathfrak{p}'$ of $\mathcal{O}_K$ dividing p' with residue field $\kappa(\mathfrak{p}') = \kappa(p') = \mathbb{F}_{p'}$.

b) The field K is not contained in $\mathbb{Q}(\zeta_m)$ for any m.

(In more elementary terms: There is an irreducible polynomial $f(X) \in \mathbb{Z}[X]$ such that whether $f(X)$ has a root modulo p depends only on $p \mod n$ for some $n \geqslant 1$, but the splitting field of $f(X)$ is not contained in $\mathbb{Q}(\zeta_m)$ for any m.)

Problem 5. Let $L|K$ be a degree n extension of number fields. Assume that its Galois closure M has Galois group $\mathrm{Gal}(M|K) \cong S_n$.

a) What is the density of prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ (ordered by norm) that split completely in $\mathcal{O}_L$?

b) What is the density of prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ (ordered by norm) that remain inert in $\mathcal{O}_L$?